Finite Generation of Families of Structures Equipped with Compatible Group Actions

by

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Contents

Biographical sketch

Charlotte Aten took courses as a nonmatriculated student at Genesee Community College, Monroe Community College, and SUNY Geneseo during the years 2006–2009. She subsequently returned to Monroe Community College, where she was awarded an Associate of Science in Mathematics degree in 2014. While at Monroe Community College she was awarded the Jan Z. Wiranowski Renaissance Scholarship and a Xerox STEM Scholarship.

After this she joined the University of Rochester as a transfer student, obtaining a Bachelor's of Science (Honors) in Mathematics in 2017. As an undergraduate she was a McNair Scholar and received the University's Doris Ermine Smith Award.

She remained at the University of Rochester for graduate school, which she began with a Provost's Fellowship for the years 2017–2019. She then held a teaching assistantship from 2019 until 2022.

Works published or in review

- **Charlotte Aten** and Alex Iosevich. "A multi-linear geometric estimate". In: *arXiv e-prints* (Dec. 2021). arXiv: `2112.00810 [math.NT]`
- **Charlotte Aten** and Semin Yoo. "Orientable smooth manifolds are essentially quasigroups". In: *arXiv e-prints* (Oct. 2021). arXiv: `2110.05660 [math.RA]`
- **Charlotte Aten**. "Multiplayer rock-paper-scissors". In: *Algebra universalis* 81.3 (2020)
- **Charlotte Aten**. "Multiplayer Rock-Paper-Scissors". In: *Algebras and Lattices in Hawai'i*. Ed. by Kira Adaricheva, William DeMeo, and Jennifer Hyndman. Apr. 2018, pp. 12–19
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Abstract

The recent theory of FI-modules provides a framework for understanding previously disparate results about the stabilization of certain infinite families of symmetric group representations indexed by the natural numbers. Here we give a different framework where the symmetric groups may be replaced by more general groups and the poset of natural numbers may be replaced by an indexing category, at the expense of an additional assumption that the group actions in question are two-sided. We prove a Noetherianess result analogous to an essential theorem for FI-modules. We then give an alternative proof for symmetric group actions which requires less representation theory than the original one for FI-modules. Finally we examine these results in relation to a notion of symmetric polynomial ideals for finite structures, generalizing a classical result of Hilbert in invariant theory. An appendix includes further technical detail on these structures, which are a categorification of Bourbaki's foundational concept.

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This thesis is an independent work.

Chapter 1

Introduction

Mathematicians are often beset by two competing impulses: the desire for concrete, finite calculations on the one hand and the allure of reasoning by general abstraction on the other. In the former case, the expression «Calculation is the path to truth.» is idiomatic, while in the latter, the ascendance of category theory during the latter half of the twentieth century is emblematic.

A guiding light in this thesis is a proposed middle path between these two ideals. If we concern ourselves not with specific examples, but rather with the manner in which those examples are constructed, then we will be able to say much more than either brute force computation or abstract nonsense would be able to tell us.

To this end we address two different mathematical stories. The first is very recent, as it is about the theory of FI-modules and mostly takes place during the 2010s. The second is older, primarily concerning events which took place during the middle of the twentieth century surrounding Bourbaki's early work on the foundations of mathematics.

In the early 2010s Church and Farb introduced the notion of an FI-module as a tool for working with the related phenomena of representation stability and homological stability in a systematic manner[8]. This came at a time when numerous examples of such phenomena were known but there was no common framework for proving that some particular system of representations or homology groups would have a consistent labeling for sufficiently large parameters. For example, as described in Farb's excellent introduction to the topic[12], it had been known for some time that when $n \geq 2$ we have that

$$H^1(\mathrm{Conf}_n(\mathbb{C}); \mathbb{C}) \cong \mathbb{C}^{\binom{n}{2}}.$$

That is, the dimension of the first cohomology group of the space encoding configurations of n points in the complex plane goes to infinity as $n \to \infty$. This is the macroscopic picture, a description of asymptotic growth in dimension which tells us little more than the rate n^2. Since the symmetric group $\mathbf{\Sigma}_n$ acts on $\mathrm{Conf}_n(\mathbb{C})$ by permuting the n points we have an induced action of $\mathbf{\Sigma}_n$ on $H^1(\mathrm{Conf}_n(\mathbb{C});\mathbb{C})$. The microscopic picture is that for any given $n \in \mathbb{N}$ we could explicitly decompose $H^1(\mathrm{Conf}_n(\mathbb{C});\mathbb{C})$ into a direct sum of irreducible representations. The significant observation here is that when $n \geq 4$ we have that

$$H^1(\mathrm{Conf}_n(\mathbb{C});\mathbb{C}) = V(0) \oplus V(1) \oplus V(2)$$

where the $V(k)$ are representations induced from those corresponding to the partitions (0), (1), and (2). This is an example of representation stability and of a result which emphasizes the manner in which the representations in question are constructed.

In [8] Church and Farb prove that this stabilization in the names of the irreducible representations comprising $H^i(\mathrm{Conf}_n(\mathbb{C});\mathbb{C})$ as a $\mathbf{\Sigma}_n$ representation occurs for each i. Their technique was further developed in [7] and [9] where they show that in many favorable cases sequences of homology groups or other representations of the symmetric groups $\mathbf{\Sigma}_n$ may be viewed as *FI-modules*, which are functors from the category FI of finite sets with injections as morphisms into a category $\mathbf{Mod}(\mathbf{R})$ of modules over a commutative unital ring $\mathbf{R}$. Moreover, many of these FI-modules of interest happen to be sub-FI-modules of ones which are easily shown to be finitely generated. In [7] it is shown that FI-modules have a *Noetherian property*, which is to say that sub-FI-modules of finitely generated FI-modules are themselves finitely generated. This provides a common framework for proving results about homological stability and representation stability.

Since many purely combinatorial quantities may be encoded by appropriate homology theories or relevant modules, these applications were quickly applied to the study of families of finite combinatorial structures acted upon by the symmetric groups. For example, in 2019 Ramos and White used these techniques to prove that polynomial formulas exist for certain combinatorial quantities associated to sequences of *FI-graphs*, which are functors from the category FI to the category **Grph** of graphs[15]. In particular, they were able to prove that for those FI-graphs $G_\bullet$ they identified as *vertex-stable* the function

$$n \mapsto \dim_{\mathbb{R}}(H_i(\mathrm{HoCo}(T, G_n);\mathbb{R}))$$

where T is a fixed graph and $\mathrm{HoCo}(T, G_n)$ is the Hom-complex of multi-homomorphisms of T into G_n eventually agrees with a polynomial of degree at most $|V(T)|\, d(i+1)$ where d is the *stable degree* of the vertex-stable FI-graph $G_\bullet$.

In 1978 Lovász used similar spaces to Hom-complexes to resolve Kneser's 1955 conjecture that whenever the n-sets in $[2n+k]$ are partitioned into $k+1$ classes there exist two disjoint subsets which belong to the same class. This can be used to determine the chromatic number of Kneser graphs[**godsil**]. This inspired work on Hom-complexes, which included Babson and Koslov showing that the spaces Lovász used were in fact Hom-complexes and Dotchermann showing that every simplicial complex can be realized as the Hom-complex of some pair of graphs.

This business of showing polynomial counts for such combinatorial quantities is our segue to an even older and more foundational discussion. When Bourbaki began writing the *Éléments de mathématique*, well before category theory had been introduced in algebraic topology, much less in the rest of mathematics, they sought to lay out in the first text of the series, *Theory of Sets*[1] a systematic description of mathematical structures as they would appear throughout the rest of the series. A simplified version of their treatment was that a *structure* was a set, say A, equipped with an indexed family $\{f_i\}_{i\in I}$ of *relations* f_i where each f_i was a subset of a set which could be constructed from A by taking Cartesian products and powersets finitely many times. Thus, denoting by $\mathrm{Sb}(A)$ the collection of subsets of A, a relation on A might be a subset of

$$A \times \mathrm{Sb}(\mathrm{Sb}(A) \times A^{57}) \times \mathrm{Sb}(\mathrm{Sb}(\mathrm{Sb}(A))),$$

for instance. Note that the now-usual relational structures of model theory are precisely these without allowing the powerset operator.

Bourbaki defined what we would now call morphisms of these structures and proved several results about them, all of which turned out to be of a categorical nature. This is only natural, since Eilenberg was a member of the group. Once his work with Mac Lane had established category theory Grothendieck and then Cartier were asked to produce a category theory component for the *Éléments*, although if either did their contribution never made it into the texts. Discussions in «La Tribu» during the 1950s seem to indicate that Bourbaki felt much of the *Éléments* would have to be rewritten in order to accommodate the new notions from category theory. More damning for categories in the *Éléments* was the difficulty of synthesizing the structural and categorical viewpoints together. The consensus became that this task was not worth the effort[11, p.328].

Returning for a moment to the more recent of our two stories, we develop in this thesis a somewhat more general theory which parallels that of FI-modules. Instead of a sequence of representations $\{\mathbf{V}_n\}_{n\in\mathbb{N}}$ of the symmetric groups $\{\mathbf{G}_n\}_{n\in\mathbb{N}}$ indexed by the category FI of finite sets with inclusions as morphisms, we consider *synergies*, which are functors from an indexing (or shape) category **S** to the category of groups, and then consider *synergy bimodules* where the component groups $\mathbf{G}_s$ have a two-sided action on the relevant modules. The two-sidedness of this action is exploited in two of our three major results on this subject. The third is more general and does apply directly to FI-modules as well, which are special cases of synergy bimodules where the synergy in question is a functor from the poset **N** of natural numbers to **Grp**.

While we don't comment on whether it would have been worth it for Bourbaki to include a fusion of the notions of category and structure in the *Éléments*, we do present one possible categorification of the concept of structure here. The formal development is mostly contained in an appendix, but the main body of this work concludes with a proof of a generalization of a result of Hilbert about symmetric polynomials[14, p.191] to the setting of finite structures. This generalization has the perhaps surprising implication that any first-order property of a finite structure **A** can be checked by counting the number of embeddings of small substructures $\mathbf{B} \hookrightarrow \mathbf{A}$, where «small» is a function of the logical complexity of the first-order property.

The remainder of this thesis is organized as follows. In chapter 2 we generalize the basic notions of FI-module theory in order to prove three main results. Our Proposition 3 gives us examples of augmentation modules which are finitely generated, while our Theorem 1 is a Noetherianess result indicating when submodules of a finitely generated **G**-bimodule are finitely generated. We close out the chapter with Theorem 2, in which we show that all submodules of a singly generated symmetric synergy bimodule are finitely generated.

In chapter 3 we give an informal overview of the notion of a structure, develop the relevant language for discussing symmetric polynomials for a given class of structures, and then prove our generalization of Hilbert's result on symmetric polynomials, which is Theorem 3. This is followed by a discussion of when the elementary symmetric polynomials generate the algebra of all symmetric polynomials freely.

Our chapter 4 consists of another summary of the results in this thesis, this time with an eye towards remaining questions and possible future work.

After the Bibliography we have Appendix A, which gives the formal treatment of structures we adjourned in chapter 3. We formally define structures in their full generality and conclude with Proposition 12, which is a structural analogue of the Yoneda Lemma and indicates the relationship between more general structures and the relational structures of model theory.

Chapter 2

Synergies and bimodules

In this chapter we develop a framework along the lines of that introduced by Church and Farb[8] for sequences of compatible actions of the symmetric groups $\mathbf{\Sigma}_n$ where the sequence of groups $\{\mathbf{\Sigma}_n\}_{n\in\mathbb{N}}$ may be replaced by a sequence of groups $\{\mathbf{G}_s\}_{s\in S}$ indexed over the objects of a small category $\mathbf{S}$ in a manner which is compatible with the morphisms of $\mathbf{S}$.

Unlike in the case of FI-module theory we assume that the actions of our groups are all two-sided. This, along with a number of assumptions on finiteness and normality of relevant generating objects for the groups $\mathbf{G}_s$ and the category $\mathbf{S}$ allows us to prove a Noetherianess result about such sequences of actions which parallels a fundamental result in that theory.

In an effort to make the litany of new notions digestible we give several examples throughout this chapter. The most basic of these is a sequence of $\mathbf{\Sigma}_n$ representations $\mathbf{V}_n$, which serve to indicate how our treatment differs from and parallels the existing theory.

2.1 Generalizing the setup for FI-module theory

We denote by $\mathbf{N} := (\mathbb{N}, \leq)$ the poset of natural numbers. Throughout this chapter we consider a small category $\mathbf{S}$ whose objects form the set S. Given $s \in S$ we denote by ι_s (or just ι) the identity morphism of s in $\mathbf{S}$.

This category $\mathbf{S}$ generalizes the role that the poset $\mathbf{N}$ plays in the theory of FI-modules, which is indexing the family of modules under consideration. We next introduce language for a family of groups indexed compatibly by the category $\mathbf{S}$.

Definition 1 (Synergy)**.** We refer to a functor $\mathbf{G}\colon\mathbf{S}\to\mathbf{Grp}$ as a *synergy* of shape $\mathbf{S}$ or as an $\mathbf{S}$*-synergy.*

For $s\in S$ we typically write $\mathbf{G}_s$ rather than $\mathbf{G}(s)$ and given a morphism $f\colon s_1\to s_2$ in $\mathbf{S}$ we simply write $\breve{f}$ rather than $\mathbf{G}(f)$.

Many familiar families of groups form synergies.

Example 1 (Symmetric synergy)*.* The *symmetric synergy* $\mathbf{\Sigma}\colon\mathbf{N}\to\mathbf{Grp}$ has for $\mathbf{\Sigma}_n$ the symmetric group of permutations of $[n]=\{1,2,\ldots,n\}$. The inclusion morphism $m\to n$ in $\mathbf{N}$ induces the canonical inclusion of $\mathbf{\Sigma}_m$ into $\mathbf{\Sigma}_n$ as permutations of $[n]$ which fix all $i>m$.

A similar example comes from the family of alternating groups.

Example 2 (Alternating synergy)*.* The *alternating synergy* $\mathbf{A}\colon\mathbf{N}\to\mathbf{Grp}$ has for $\mathbf{A}_n$ the alternating group on $[n]$. The inclusion morphisms are as in $\mathbf{\Sigma}$.

We can consider the indexing category $\mathbf{S}$ to be a more involved poset as well. Let $\mathbf{N}^2$ denote the direct product of the poset $\mathbf{N}$ with itself, viewed as a category.

Example 3 (General linear synergy)*.* Fix a field $\mathbb{F}$. The *general linear synergy* $\mathbf{GL}(\mathbb{F})\colon\mathbf{N}^2\to\mathbf{Grp}$ has

$$(\mathbf{GL}(\mathbb{F}))_{i,j}\coloneqq\mathbf{GL}_{i+j}(\mathbb{F}),$$

the group of invertible square matrices of size $i+j$ over $\mathbb{F}$. The inclusion morphisms are generated by the images of $(i,j)\to(i+1,j)$ and $(i,j)\to(i,j+1)$, include a matrix A as a block

$$\begin{bmatrix}1&0\\0&A\end{bmatrix}\text{ and }\begin{bmatrix}A&0\\0&1\end{bmatrix},$$

respectively.

In order to codify a compatible action of the groups $\mathbf{G}_s$ we make use of the following auxiliary category built from $\mathbf{G}$.

Definition 2 (Unspooling of a synergy)**.** Given an $\mathbf{S}$-synergy $\mathbf{G}$ the *unspooling* of $\mathbf{G}$ is the category $\mathcal{G}$ whose objects are the elements of S, whose morphism sets are

$$\mathrm{Hom}_{\mathcal{G}}(s_1,s_2)\coloneqq\{\,\sigma f\tau\mid\sigma,\tau\in G_{s_2}\text{ and }f\colon s_1\to s_2\,\}\,,$$

whose composition map

$$\circ\colon \mathrm{Hom}_{\mathcal{G}}(s_2, s_3) \times \mathrm{Hom}_{\mathcal{G}}(s_1, s_2) \to \mathrm{Hom}_{\mathcal{G}}(s_1, s_3)$$

is given by

$$(\sigma_3 g \tau_3) \circ (\sigma_2 f \tau_2) = \sigma_3 \breve{g}(\sigma_2)(g \circ f)\breve{g}(\tau_2)\tau_3,$$

and whose identity morphisms are those of the form $e\iota e$.

We usually omit identity elements, so $e\iota e = \iota e = e\iota = \iota$ as morphisms in $\mathcal{G}$.

Example 4 (The unspooling of the symmetric synergy)*.* When $\mathbf{G}$ is the symmetric synergy $\mathbf{\Sigma}$ the unspooling $\mathcal{G}$ will take on a similar role for us that the category FI has in the theory of FI-modules. The category $\mathcal{G}$ is not the same as FI in this case, however.

Definition 3 (Synergy biobject)**.** Given a synergy $\mathbf{G}$ and a category $\mathscr{C}$ we refer to a functor $\mathbf{V}\colon \mathcal{G} \to \mathscr{C}$ as a $\mathbf{G}$-*biobject in* $\mathscr{C}$.

When objects in the category $\mathscr{C}$ are called «blahs» we also refer to $\mathbf{G}$-biobjects as $\mathbf{G}$-*biblahs*.

Example 5 (Symmetric synergy biset)*.* Let $\mathbf{G}$ be the symmetric synergy $\mathbf{\Sigma}$ and consider the biset $V\colon \mathcal{G} \to \mathscr{C}$ where $V(n) \coloneqq \Sigma_n$, the set of all permutations of $[n]$, and when $f\colon m \to n$ is a morphism in $\mathbf{N}$ and $\sigma, \tau \in \Sigma_n$ we define

$$V(\sigma f \tau)\colon \Sigma_m \to \Sigma_n$$

by

$$(V(\sigma f \tau))(\upsilon) \coloneqq \sigma \upsilon \tau$$

where υ is viewed as a permutation of n which fixes all $i > m$.

We can produce similar examples where we eliminate either the left or right action of the symmetric groups by replacing $\sigma\upsilon\tau$ with $\sigma\upsilon$ or $\upsilon\tau$ in the preceding example.

Definition 4 (Synergy category)**.** Given a synergy $\mathbf{G}$ and a category $\mathscr{C}$ we refer to the functor category $\mathbf{G}\mathscr{C} \coloneqq \mathbf{Fun}(\mathcal{G}, \mathscr{C})$ as the *category of* $\mathbf{G}$-*biobjects in* $\mathscr{C}$.

Note that the objects of $\mathbf{G}\mathscr{C}$ are the $\mathbf{G}$-biobjects in $\mathscr{C}$. We will be particularly interested in the case that $\mathscr{C} = \mathbf{Mod}(\mathbf{R})$ is the category of modules over a commutative unital ring $\mathbf{R}$. Observe that for any such choice of $\mathbf{R}$ and any $\mathbf{G}$ we have that $\mathbf{G}\,\mathbf{Mod}(\mathbf{R})$ is an abelian category[16, p.25].

When we have a $\mathbf{G}$-bimodule $\mathbf{V}\colon \mathcal{G} \to \mathbf{Mod}(\mathbf{R})$ and some morphism $\sigma f \tau$ in $\mathcal{G}$ we often write $\overline{\sigma f \tau}$ rather than $\mathbf{V}(\sigma f \tau)$. Similarly, $v \in V_s$ we write $\overline{\sigma} v$ rather than $\mathbf{V}(\sigma\iota)(v)$ and $v\overline{\tau}$ rather than $\mathbf{V}(\iota\tau)(v)$.

The notation $\bar{\sigma} v \bar{\tau}$ is actually not ambiguous, although one might first worry that it is given the competing notations introduced in the previous paragraph. One may read $\bar{\sigma} v \bar{\tau}$ as either $\mathbf{V}(\iota\tau)(\mathbf{V}(\sigma\iota)(v))$ or $\mathbf{V}(\sigma\iota)(\mathbf{V}(\iota\tau)(v))$, but we have that

$$\begin{aligned}
\mathbf{V}(\iota\tau)(\mathbf{V}(\sigma\iota)(v)) &= (\mathbf{V}(e\iota\tau) \circ \mathbf{V}(\sigma\iota e))(v) \\
&= (\mathbf{V}(e\iota\tau \circ \sigma\iota e))(v) \\
&= (\mathbf{V}(e\breve{\iota}(\sigma)\iota\breve{\iota}(e)\tau))(v) \\
&= (\mathbf{V}(\sigma\iota\tau))(v) \\
&= (\mathbf{V}(\sigma\breve{\iota}(e)\iota\breve{\iota}(\tau)e))(v) \\
&= (\mathbf{V}(\sigma\iota e \circ e\iota\tau))(v) \\
&= (\mathbf{V}(\sigma\iota e) \circ \mathbf{V}(e\iota\tau))(v) \\
&= \mathbf{V}(\sigma\iota)(\mathbf{V}(\iota\tau)(v)).
\end{aligned}$$

Another way to say this is that $\bar{\sigma} v \bar{\tau} = \overline{\sigma\iota\tau}(v)$. Yet another is that each $\mathbf{V}_s$ is a bimodule over $\mathbf{RG}_s$, making our language consistent with existing usage.

Definition 5 (Synergy bimodule category)**.** Given a commutative unital ring $\mathbf{R}$ and a synergy $\mathbf{G}$ we refer to $\mathbf{G}\,\mathbf{Mod}(\mathbf{R})$ as the *category of* $\mathbf{G}$*-bimodules (over* $\mathbf{R}$*).*

In our theory of synergy bimodules functors from $\mathbf{S}$ to $\mathbf{Set}$ play a role analogous to that of generating sets for a group bimodule.

Definition 6 (Category set)**.** We refer to a functor $\Psi\colon \mathbf{S} \to \mathbf{Set}$ as a *category set* or as an $\mathbf{S}$*-set.*

As we did with synergies we generally write Ψ_s rather than $\Psi(s)$ and $\breve{f}$ rather than $\Psi(f)$. We have an alternative notation for the category of $\mathbf{S}$-sets.

Definition 7 (Category of category sets)**.** We define $\mathbf{S}\,\mathbf{Set} \coloneqq \mathbf{Fun}(\mathbf{S}, \mathbf{Set})$ to be the *category of* $\mathbf{S}$*-sets.*

We need notions of generation and finiteness for $\mathbf{S}$-sets.

Definition 8 (Generating set of a category set)**.** Given an $\mathbf{S}$-set $\Psi\colon \mathbf{S} \to \mathbf{Set}$ we say that $\Psi' \subset \bigcup_{s \in S} \Psi_s$ is a *generating set* for Ψ when for each $s_2 \in S$ we have that

$$\Psi_{s_2} = \bigcup_{f\colon s_1 \to s_2} \breve{f}(\Psi' \cap \Psi_{s_1}).$$

Definition 9 (Finite category set)**.** We say that an $\mathbf{S}$-set $\Psi\colon \mathbf{S} \to \mathbf{Set}$ is *finite* when Ψ has a finite generating set.

Definition 10 (Base of a category set)**.** Given a category set Ψ with generating set $\Psi' \subset \bigcup_{s \in S} \Psi_s$ we refer to

$$B \coloneqq \{\, s \in S \mid \Psi' \cap \Psi_s \neq \varnothing \,\}$$

as the *base* of Ψ associated to Ψ'.

Definition 11 (Order of a finite category set)**.** When $\Psi\colon \mathbf{S} \to \mathbf{Set}$ is finite we say that the *order* (or *size*) of Ψ is the minimum size of a finite generating set for Ψ. We denote the order of Ψ by $|\Psi|$.

Note that if Ψ is finite we have $|\Psi| \geq |B|$ where B is a base associated to a minimum size finite generating set for Ψ.

We have an adjunction between the category $\mathbf{G}\,\mathbf{Mod}(\mathbf{R})$ and the category $\mathbf{S}\,\mathbf{Set}$.

Definition 12 (Forgetful functor)**.** We refer to the functor

$$\mathrm{For}\colon \mathbf{G}\,\mathbf{Mod}(\mathbf{R}) \to \mathbf{S}\,\mathbf{Set}$$

given by

$$(\mathrm{For}(\mathbf{V}))_s \coloneqq V_s$$

and

$$(\mathrm{For}(\eta))_s \coloneqq \eta_s\colon (V_1)_s \to (V_2)_s$$

where $\mathbf{V}$ is a $\mathbf{G}$-bimodule and $\eta\colon \mathbf{V}_1 \to \mathbf{V}_2$ is a morphism of $\mathbf{G}$-bimodules as the *forgetful functor* from $\mathbf{G}\,\mathbf{Mod}(\mathbf{R})$ to $\mathbf{S}\,\mathbf{Set}$.

This functor has a left adjoint, which we explicitly describe.

Definition 13 (Free functor)**.** We refer to the functor

$$\mathbf{Fr}\colon \mathbf{S}\,\mathbf{Set} \to \mathbf{G}\,\mathbf{Mod}(\mathbf{R})$$

given by

$$(\mathbf{Fr}(\Psi))_s \coloneqq \mathbf{R}[\{\, \sigma\psi\tau \mid \psi \in \Psi_s \text{ and } \sigma, \tau \in G_s \,\}]$$

with

$$\overline{\sigma_2 f \tau_2}(\sigma_1 \psi \tau_1) \coloneqq \sigma_2 \breve{f}(\sigma_1)\breve{f}(\psi)\breve{f}(\tau_1)\tau_2.$$

where Ψ is an $\mathbf{S}$-set, $\eta\colon \Psi_1 \to \Psi_2$ is a morphism of $\mathbf{P}$-sets, and

$$(\mathbf{Fr}(\eta))_s\colon (\mathbf{Fr}(\Psi_1)) \to (\mathbf{Fr}(\Psi_2))_s$$

is the $\mathbf{R}$-linear extension of the set map $\eta_s\colon (\Psi_1)_s \to (\Psi_2)_s$ as the *free functor* from $\mathbf{S}\,\mathbf{Set}$ to $\mathbf{G}\,\mathbf{Mod}(\mathbf{R})$.

Thus, there is a natural isomorphism

$$\mathbf{G}\,\mathbf{Mod}(\mathbf{R})(\mathbf{Fr}(\Psi), \mathbf{V}) \cong \mathbf{S}\,\mathbf{Set}(\Psi, \mathrm{For}(\mathbf{V})).$$

We have a similar object which plays the same role as the regular representation in classical representation theory.

Definition 14 (Regular synergy bimodule)**.** Given an $\mathbf{S}$-synergy $\mathbf{G}$, a unital commutative ring $\mathbf{R}$, and an $\mathbf{S}$-set Ψ we define the *regular* $\mathbf{G}$*-bimodule*

$$\mathbf{RG}[\Psi]\colon \mathcal{G} \to \mathbf{Mod}(\mathbf{R})$$

by

$$(\mathbf{RG}[\Psi])_s \coloneqq \mathbf{R}[\{\, \sigma\psi \mid \psi \in \Psi_s \text{ and } \sigma \in G_s \,\}]$$

and

$$\overline{\sigma_2 f \tau_2}(\sigma_1 \psi) \coloneqq \sigma_2 \breve{f}(\sigma_1)\tau_2 \breve{f}(\psi).$$

We also have a notion of finite generation for $\mathbf{G}$-bimodules.

Definition 15 (Finitely generated synergy bimodule)**.** We say that a $\mathbf{G}$-bimodule $\mathbf{V}\colon \mathcal{G} \to \mathbf{Mod}(\mathbf{R})$ is *finitely generated* when there exists an epimorphism $\mathbf{Fr}(\Psi) \twoheadrightarrow \mathbf{V}$ where Ψ is finite.

A finitely generated synergy bimodule is thus determined by elements lying in a certain collection of modules $\mathbf{V}_s$.

We need notions analogous to those of group coinvariants for $\mathbf{G}$-bimodules.

Definition 16 (Augmentation ideal)**.** Given a **G**-bimodule $\mathbf{V}\colon \mathcal{G} \to \mathbf{Mod}(\mathbf{R})$ the *augmentation ideal* $\mathbf{\Theta V}\colon \mathcal{G} \to \mathbf{Mod}(\mathbf{R})$ is the sub-**G**-bimodule of **V** with $(\mathbf{\Theta V})_s$ defined to be the sub-**R**-module of $\mathbf{V}_s$ generated by

$$\{\, v - \bar{\sigma} v \bar{\tau} \mid v \in V_s,\, \sigma, \tau \in G_s \,\}\,.$$

The following proposition shows that $\mathbf{\Theta V}$ does have appropriately restricted inclusion homomorphisms, which are necessary for it to be a sub-**G**-bimodule.

Proposition 1. *For any $f\colon s_1 \to s_2$ we have that $\bar{f}(\Theta V)_{s_1} \subset (\Theta V)_{s_2}$.*

Proof. Note that for any $v \in V_{s_1}$ and any $\sigma, \tau \in G_{s_2}$ we have that

$$\bar{f}(\bar{\sigma} v \bar{\tau}) = \overline{\breve{f}(\sigma)}\bar{f}(v)\overline{\breve{f}(\tau)}.$$

Since $\bar{f}$ is a homomorphism of modules we have for any $v \in V_{s_1}$ and any $\sigma, \tau \in G_{s_2}$ that

$$\bar{f}(v - \bar{\sigma} v \bar{\tau}) = \bar{f}(v) - \bar{f}(\bar{\sigma} v \bar{\tau})$$

and hence

$$\bar{f}(v - \bar{\sigma} v \bar{\tau}) = \bar{f}(v) - \overline{\breve{f}(\sigma)}\bar{f}(v)\overline{\breve{f}(\tau)}.$$

This shows that $\bar{f}$ takes generators of $(\mathbf{\Theta V})_{s_1}$ (as an $\mathbf{RG}_{s_1}$-bimodule) to generators of $(\mathbf{\Theta V})_{s_2}$ (as an $\mathbf{RG}_{s_2}$-bimodule). This is not yet enough for our purposes since we have only noted that $\bar{f}$ is an **R**-module homomorphism. Fortunately we have for any $\sigma_1, \sigma_2, \tau_1, \tau_2 \in G_{s_1}$ that

$$\begin{aligned}\bar{f}(\bar{\sigma}_2(v - \bar{\sigma}_1 v \bar{\tau}_1)\bar{\tau}_2) &= \overline{\breve{f}(\sigma_2)}\bar{f}(v - \bar{\sigma}_1 v \bar{\tau}_1)\overline{\breve{f}(\tau_2)} \\ &= \overline{\breve{f}(\sigma_2)}(\bar{f}(v) - \overline{\breve{f}(\sigma_1)}\bar{f}(v)\overline{\breve{f}(\tau_1)})\overline{\breve{f}(\tau_2)},\end{aligned}$$

so $\bar{f}$ takes the generators of $(\mathbf{\Theta V})_{s_1}$ (as an **R**-module) to generators of $(\mathbf{\Theta V})_{s_2}$ (as an **R**-module), which suffices to prove the claim. □

If we assume some additional structure on the shape category **S** we may associate to each **G**-bimodule a graded module over a suitable ring.

Definition 17 (Escalation)**.** Given a category **S** and an endofunctor $\mathring{\xi}\colon \mathbf{S} \to \mathbf{S}$ we refer to a natural transformation $\xi\colon \mathrm{id}_{\mathbf{S}} \to \mathring{\xi}$ as an *escalation* of **S**.

Definition 18 (Class of escalations)**.** We denote by $\mathrm{Esc}(\mathbf{S})$ the class of all escalations of a category $\mathbf{S}$.

Escalations of a category generalize both isotone maps from a poset to itself as well as inner automorphisms of a group. It is worth noting that in the group case this formulation is reminiscent of the automorphisms considered by Cohen et al.[10] in their analysis of the cohomology of the pure braid permutation group.

Example 6 (Poset escalations are isotone maps). Let $\mathbf{P}$ be a poset viewed as a category where there is exactly one morphism $f_{a,b}\colon a \to b$ whenever $a \leq b$ in $\mathbf{P}$. The set $\mathrm{Esc}(\mathbf{P})$ may be identified with the set of all isotone maps from $\mathbf{P}$ to itself. To see this, note that each endofunctor $\mathring{\xi}\colon \mathbf{P} \to \mathbf{P}$ is an isotone map. A natural transformation $\xi\colon \mathrm{id}_{\mathbf{P}} \to \mathring{\xi}$ has components of the form $\xi_a\colon a \to \mathring{\xi}(a)$. Since these components are morphisms in $\mathbf{P}$ they indicate precisely that each a lies below $\mathring{\xi}(a)$.

Example 7 (Group escalations are inner automorphisms). Let $\mathbf{G}$ be a group viewed as a category with one object, say $*$. The set $\mathrm{Esc}(\mathbf{G})$ may be identified with the set of all inner automorphisms of $\mathbf{G}$. To see this, note that each endofunctor $\mathring{\xi}\colon \mathbf{G} \to \mathbf{G}$ is a group endomorphism of $\mathbf{G}$. Given a natural transformation $\xi\colon \mathrm{id}_{\mathbf{G}} \to \mathring{\xi}$ and a morphism a in $\mathbf{G}$ (which is nothing more than an element of G) we have that $\xi_* a = \mathring{\xi}(a)\xi_*$ so the endomorphism $\mathring{\xi}$ is conjugation by the element ξ_*.

It is not a coincidence that the escalations in those two cases form monoids under composition.

Definition 19 (Escalation monoid)**.** Given a category $\mathbf{S}$ the *escalation monoid* $\mathbf{Esc}(\mathbf{S})$ is a the monoid whose elements form the class $\mathbf{S}$, whose composition is the horizontal composition of natural transformations, and whose identity is the identity natural transformation $\mathrm{id}_{\mathrm{id}_{\mathbf{S}}}\colon \mathrm{id}_{\mathbf{S}} \to \mathrm{id}_{\mathbf{S}}$ of the identity functor of $\mathbf{S}$.

Definition 20 (Generating set of a category)**.** Given a category $\mathbf{S}$, a collection $\Xi \subset \mathrm{Esc}(\mathbf{S})$, and some $B \subset S$ we say that Ξ is a *generating set* for $\mathbf{S}$ *based at* B when

1. for each $s \in S$, each $b \in B$, and each morphism $f\colon b \to s$ we can write

$$f = (\xi_k \cdots \xi_1)_b$$

   for some $\xi_1, \ldots, \xi_k \in \Xi$ and

2. each $s \in S$ is the codomain of a morphism $f\colon b \to s$ where $b \in B$.

We need a notion of Noetherianess for categories. Here a Noetherian ring is taken to be a ring which is Noetherian as a left module over itself, which is the same as taking all left ideals to be finitely generated.

Definition 21 (Escalation ring)**.** Given a category $\mathbf{S}$ and a unital commutative ring $\mathbf{R}$ we denote by $\mathbf{R}\,\mathbf{Esc}(\mathbf{S})$ the *escalation ring (of* $\mathbf{S}$ *over* $\mathbf{R}$*)*, which is the monoid ring of $\mathbf{Esc}(\mathbf{S})$ over $\mathbf{R}$.

Definition 22 (Ring of a set of escalations)**.** Given a category $\mathbf{S}$ and some $\Xi \subset \mathrm{Esc}(\mathbf{S})$ we denote by $\mathbf{R}\{\Xi\}$ the subring of $\mathbf{R}\,\mathbf{Esc}(\mathbf{S})$ generated by $R \cup \Xi$.

Definition 23 (Coinvariants module)**.** Let $\mathbf{G}$ be an $\mathbf{S}$-synergy which has a generating set Ξ and let $\mathbf{R}$ be a unital commutative ring. Given a $\mathbf{G}$-bimodule $\mathbf{V}\colon \mathcal{G} \to \mathbf{Mod}(\mathbf{R})$ the Ξ-*coinvariants module* $\mathbf{\Phi V}$ is an S-graded $\mathbf{R}\{\Xi\}$-module whose s^{th} component is

$$(\mathbf{\Phi V})_s \coloneqq \mathbf{V}_s/(\mathbf{\Theta V})_s$$

and for which $\xi \in \Xi$ acts as a map

$$\dot{\xi}_s\colon (\mathbf{\Phi V})_s \to (\mathbf{\Phi V})_{\mathring{\xi}(s)}$$

which is given by

$$\dot{\xi}_s(v/(\Theta V)_s) \coloneqq \bar{\xi}_s(v)/(\Theta V)_{\mathring{\xi}(s)}.$$

We verify that the maps $\dot{\xi}_s$ are well-defined.

Proposition 2. *We have that the map $\dot{\xi}_s$ is well-defined. That is, given $v_1, v_2 \in V_s$ with $v_1/(\Theta V)_s = v_2/(\Theta V)_s$ we have that*

$$\bar{\xi}_s(v_1)/(\Theta V)_{\mathring{\xi}(s)} = \bar{\xi}_s(v_2)/(\Theta V)_{\mathring{\xi}(s)}.$$

Proof. Since $v_1/(\Theta V)_s = v_2/(\Theta V)_s$ we know that $v_1 - v_2 \in (\Theta V)_s$. Since $\bar{\xi}_s$ is a homomorphism of $\mathbf{R}$-modules we have that

$$\bar{\xi}_s(v_1) - \bar{\xi}_s(v_2) \in \bar{\xi}_s((\Theta V)_s) \subset (\Theta V)_{\mathring{\xi}(s)},$$

as desired. □

Definition 24 (Noetherian category)**.** Given a category $\mathbf{S}$ which is finitely generated by (Ξ, B) and a unital commutative ring $\mathbf{R}$ we say that $\mathbf{S}$ is $(\mathbf{R}, \Xi)$-*Noetherian* (or *Noetherian (over* $\mathbf{R}$ *with respect to* Ξ*)*) when $\mathbf{R}\{\Xi\}$ is a Noetherian ring.

2.2 Finite generation of the augmentation ideal

We proceed to the notion of a finitely-generated synergy with respect to conjugation.

Definition 25 (Normal generating set for a synergy)**.** Given an $\mathbf{S}$-synergy $\mathbf{G}$ we say that (Ω, Ω', B) is a *normal generating set* for $\mathbf{G}$ when $\Omega \leq \mathrm{For}(\mathbf{G})$ has a generating set Ω' whose associated base is B such that for each $s_1 \in B$ and each morphism $f\colon s_1 \to s_2$ in $\mathbf{S}$ we have that

$$G_{s_2} = \mathrm{Nml}\left(\left\{\, \breve{f}(\omega) \,\middle|\, \omega \in \Omega' \cap \Omega_{s_1} \,\right\}\right).$$

Definition 26 (NFG synergy)**.** When $\mathbf{G}$ is a synergy with a normal generating set (Ω, Ω', B) where Ω' is finite we say that $\mathbf{G}$ is an *NFG synergy* or that $\mathbf{G}$ *is NFG (by* (Ω, Ω', B)*)*.

NFG synergies have well-behaved regular synergy bimodules.

Proposition 3. *If* $\mathbf{G}$ *is a synergy then for any finite* $\mathbf{S}$*-set* Ψ *we have that* $\mathbf{RG}[\Psi]$ *is finitely generated. If* $\mathbf{G}$ *is NFG by* (Ω, Ω', B) *and* Ψ *is finite with finite generating set* Ψ' *whose associated base is* B *then* $\mathbf{\Theta G}[\Psi]$ *is finitely generated.*

Proof. Consider the morphism of $\mathbf{G}$-bimodules $\mathbf{Fr}(\Psi) \to \mathbf{RG}[\Psi]$ given by $\psi \mapsto \psi$. Since this map is epic and Ψ is finite we have that $\mathbf{RG}[\Psi]$ is finitely generated.

In the case of $\mathbf{\Theta G}[\Psi]$ for a synergy $\mathbf{G}$ which is NFG by (Ω, Ω', B) we define

$$(\Psi')^{\Omega'} \coloneqq \bigcup_{s \in B} \left\{\, \psi - \omega^{\pm 1}\psi \,\middle|\, \psi \in \Psi' \cap \Psi_s \text{ and } \omega \in \Omega' \cap \Omega_s \,\right\}$$

and

$$\Psi^{\Omega}\colon \mathbf{S} \to \mathbf{Set}$$

where

$$\Psi^{\Omega}_{s_2} \coloneqq \bigcup_{\substack{s_1 \in B \\ f\colon s_1 \to s_2}} \left\{\, \breve{f}(\psi) - \breve{f}(\omega^{\pm 1})\breve{f}(\psi) \,\middle|\, \psi \in \Psi' \cap \Psi_{s_1} \text{ and } \omega \in \Omega' \cap \Omega_{s_1} \,\right\}$$

and when $f_3\colon s_2 \to s_3$ we set

$$\breve{f}_3(\breve{f}_2(\psi) - \breve{f}_2(\omega^{\pm 1})\breve{f}_2(\psi)) \coloneqq \breve{f}(\psi) - \breve{f}(\omega^{\pm 1})\breve{f}(\psi)$$

where $f = f_3 \circ f_2$. We claim that $(\Psi')^{\Omega'}$ is a finite generating set for Ψ^{Ω} and that the morphism $q\colon \mathbf{Fr}(\Psi^{\Omega}) \to \mathbf{\Theta G}[\Psi]$ given by

$$q(\breve{f}(\psi) - \breve{f}(\omega^{\pm 1})\breve{f}(\psi)) \coloneqq \breve{f}(\psi) - \breve{f}(\omega^{\pm 1})\breve{f}(\psi)$$

is epic.

In order to show that q is an epimorphism it suffices to show that each generator $v - \bar{\sigma} v \bar{\tau}$ of $\mathbf{\Theta G}[\Psi]$ lies in the image of q. Since $\mathbf{RG}[\Psi]$ is finitely generated we have that each $v \in (RG[\Psi])_{s_2}$ may be written as

$$v = \sum_{s_1 \in B} \sum_{f\colon s_1 \to s_2} \sum_{\psi \in \Psi' \cap \Psi_{s_1}} \alpha_{f,\psi} \breve{f}(\psi)$$

where only finitely many of the $\alpha_{f,\psi} \in RG_{s_2}$ are nonzero. By linearity it suffices to show that each $\breve{f}(\psi) - \sigma \breve{f}(\psi)$ lies in the image of q for $\sigma \in G_{s_2}$.

Since $\mathbf{G}$ is NFG we have for $s_1 \in B$ and $f\colon s_1 \to s_2$ that every $\sigma \in G_{s_2}$ may be written as

$$\sigma = \prod_{i=1}^{n} a_i \breve{f}(\omega_i^{\pm 1}) a_i^{-1}$$

where $a_i \in G_{s_2}$ and $\omega \in \Omega' \cap \Omega_{s_1}$. We argue by induction on n. For the base case, suppose that $\sigma = a\breve{f}(\omega^{\pm 1})a^{-1}$ where $a \in G_{s_2}$ and $\omega_i \in \Omega' \cap \Omega_{s_1}$. In this case we have that

$$\breve{f}(\psi) - \sigma \breve{f}(\psi) = \breve{f}(\psi) - a\breve{f}(\omega^{\pm 1})a^{-1}\breve{f}(\psi) = a(\breve{f}(\psi) - \breve{f}(\omega^{\pm 1})\breve{f}(\psi))a^{-1}.$$

Since

$$q(a(\breve{f}(\psi) - \breve{f}(\omega^{\pm 1})\breve{f}(\psi))a^{-1}) = a(\breve{f}(\psi) - \breve{f}(\omega^{\pm 1})\breve{f}(\psi))a^{-1}$$

we have that $\breve{f}(\psi) - \sigma \breve{f}(\psi)$ must belong to the image of q.

Now suppose that

$$\sigma = \prod_{i=1}^{n} a_i \breve{f}(\omega_i^{\pm 1}) a_i^{-1}$$

where $n \geq 2$ and let

$$\sigma' := \prod_{i=2}^{n} a_i \breve{f}(\omega_i^{\pm 1}) a_i^{-1}.$$

Since $\breve{f}(\psi) - \sigma' \breve{f}(\psi)$ lies in the image of q we have that

$$q(x') = \breve{f}(\psi) - \sigma' \breve{f}(\psi)$$

for some $x' \in (\mathrm{Fr}(\Psi^{\Omega}))_{s_2}$. We also have that there exists some $x \in (\mathrm{Fr}(\Psi^{\Omega}))_{s_2}$ such that

$$q(x) = \breve{f}(\psi) - a_1 \breve{f}(\omega_1^{\mp 1}) a_1^{-1} \breve{f}(\psi).$$

Observe that

$$q(a_1 \breve{f}(\omega_1^{\pm 1}) a_1^{-1}(x' - x)) = \breve{f}(\psi) - \sigma \breve{f}(\psi),$$

as desired. $\square$

Note that we get a relatively explicit bound on the size of a finite generating set for $\mathbf{\Theta G}[\Psi]$ since we have that

$$\left|\Psi^{\Omega}\right| \leq \left|(\Psi')^{\Omega'}\right| \leq 2 \sum_{s \in B} |\Psi' \cap \Psi_s| \, |\Omega' \cap \Omega_s| \, .$$

2.3 The Noetherianess of synergy bimodules

We have a Noetherianess property for certain synergy bimodules.

Theorem 1. *Suppose that $\mathbf{G}$ is an $\mathbf{S}$-synergy and that $\mathbf{V}\colon \mathcal{G} \to \mathbf{Mod}(\mathbf{R})$ is a $\mathbf{G}$-bimodule with $\mathbf{W} \leq \mathbf{V}$. If*

1. *$\mathbf{\Theta W}$ is finitely generated with witness $q_{\Theta}\colon \mathbf{Fr}(\Psi_{\Theta}) \twoheadrightarrow \mathbf{V}$ where Ψ_{Θ} is finite with finite generating set Ψ'_{Θ} whose associated base is B_{Θ},*

2. *$\mathbb{Q} \leq \mathbf{R}$,*

3. *all the groups $\mathbf{G}_s$ are torsion,*

4. *$\mathbf{S}$ is $(\mathbf{R}, \Xi)$-Noetherian,*

5. $\mathbf{V}$ *is finitely generated with witness* $q\colon \mathbf{Fr}(\Psi) \twoheadrightarrow \mathbf{V}$ *where* Ψ *is finite with finite generating set* Ψ' *whose associated base is* B*,*

6. $\mathbf{S}$ *is generated by* (Ξ, B)

then $\mathbf{W}$ *is finitely generated.*

Proof. We show that $\mathbf{W}$ is finitely generated by combining the finite generating set we assume in (1) for $\mathbf{\Theta W}$ with a finite generating set we obtain for $\mathbf{\Phi W}$ in order to produce a finite generating set for $\mathbf{W}$.

By our assumptions (2) and (3) the coinvariants functor Φ is left exact so the short exact sequence

$$0 \longrightarrow \mathbf{W} \hookrightarrow \mathbf{V} \twoheadrightarrow \mathbf{V}/\mathbf{W} \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow \mathbf{\Phi W} \hookrightarrow \mathbf{\Phi V} \longrightarrow \mathbf{\Phi V}/\mathbf{\Phi W},$$

which means, in particular, that $\mathbf{\Phi W}$ is an S-graded sub-$\mathbf{R}\{\Xi\}$-module of $\mathbf{\Phi V}$. Since our assumption (4) is that $\mathbf{S}$ is $(\mathbf{R}, \Xi)$-Noetherian we have that $\mathbf{R}\{\Xi\}$ is a Noetherian ring. If we establish that the $\mathbf{R}\{\Xi\}$-module $\mathbf{\Phi V}$ is finitely generated then we would have that $\mathbf{\Phi W}$ is a submodule of a finitely generated module over a Noetherian ring and hence is itself finitely generated.

Consider an element

$$v/(\Theta V)_{s_2} \in (\Phi V)_{s_2} = V_{s_2}/(\Theta V)_{s_2}.$$

By assumption (5) we know that $\mathbf{V}$ is generated by the $q(\psi)$ for $\psi \in \Psi$ so we can write

$$v = \sum_{s_1 \in B} \sum_{f : s_1 \to s_2} \sum_{\psi \in \Psi' \cap \Psi_{s_1}} \sum_{\sigma, \tau \in G_{s_2}} \alpha_{f,\psi,\sigma,\tau} \bar{\sigma} q(\breve{f}(\psi)) \bar{\tau}$$

where only finitely many of the $\alpha_{f,\psi,\sigma,\tau} \in R$ are nonzero. It follows that

$$v/(\Theta V)_{s_2} = \sum_{s_1 \in B} \sum_{f : s_1 \to s_2} \sum_{\psi \in \Psi' \cap \Psi_{s_1}} \sum_{\sigma, \tau \in G_{s_2}} \alpha_{f,\psi,\sigma,\tau} \bar{\sigma} q(\breve{f}(\psi)) \bar{\tau}/(\Theta V)_{s_2}$$

but since $\bar{\sigma} q(\breve{f}(\psi)) \bar{\tau}/(\Theta V)_{s_2} = q(\breve{f}(\psi))/(\Theta V)_{s_2}$ we have that

$$v/(\Theta V)_{s_2} = \sum_{s_1 \in B} \sum_{f : s_1 \to s_2} \sum_{\psi \in \Psi' \cap \Psi_{s_1}} \alpha_{f,\psi} q(\breve{f}(\psi))/(\Theta V)_{s_2}$$

where only finitely many of the $\alpha_{f,\psi} \in R$ are nonzero.

We claim that the elements $\{\, q(\psi)/(\Theta V)_s \mid s \in B \text{ and } \psi \in \Psi' \cap \Psi_s \,\}$ generate $\mathbf{\Phi V}$ as an $\mathbf{R}\{\Xi\}$-module. Observe that given $s_1 \in B$, $f\colon s_1 \to s_2$, and $\psi \in \Psi' \cap \Psi_{s_1}$ we have that

$$q(\breve{f}(\psi)) = \bar{f}(q(\psi)) = q(\dot{\xi}_k \cdots \dot{\xi}_1 \psi) = \dot{\xi}_k \cdots \dot{\xi}_1 q(\psi)$$

since by assumption (6) the category $\mathbf{S}$ is generated by (Ξ, B).

Since $\mathbf{\Phi V}$ is a finitely generated module over a Noetherian ring the submodule $\mathbf{\Phi W}$ is also finitely generated.

By assumption (1) we know that $\mathbf{\Theta W}$ is finitely generated with witness

$$q_\Theta\colon \mathbf{Fr}(\Psi_\Theta) \to \mathbf{\Theta W}$$

where Ψ_Θ is finite with finite generating set Ψ'_Θ whose associated base is B_Θ and by our preceding work we have that $\mathbf{\Phi W}$ is finitely generated with witness

$$q_\Phi\colon \mathbf{Fr}(\Psi_\Phi) \to \mathbf{\Phi W}$$

where Ψ_Φ is finite with finite generating set Ψ'_Φ whose associated base is B. Since $\mathbf{\Phi W}$ is so finitely generated we know that for each $s_2 \in S$ and each

$$w/(\Theta W)_{s_2} \in (\Phi W)_{s_2}$$

we can write $w/(\Theta W)_{s_2}$ as

$$w/(\Theta W)_{s_2} = \sum_{s_1 \in B} \sum_{f\colon s_1 \to s_2} \sum_{\psi \in \Psi'_\Phi \cap (\Psi_\Phi)_{s_1}} \alpha_{f,\psi} \bar{f}(q_\Phi(\psi))/(\Theta W)_{s_2}$$

where only finitely many of the $\alpha_{f,\psi} \in R$ are nonzero. Thus,

$$w = \sum_{s_1 \in B} \sum_{f\colon s_1 \to s_2} \sum_{\psi \in \Psi'_\Phi \cap (\Psi_\Phi)_{s_1}} \alpha_{f,\psi} \bar{f}(q_\Phi(\psi)) + y$$

for some $y \in (\Theta W)_{s_2}$.

Since $\mathbf{\Theta W}$ is finitely generated we can write

$$y = \sum_{s_1 \in B_\Theta} \sum_{f: s_1 \to s_2} \sum_{\psi \in \Psi'_\Theta \cap (\Psi_\Theta)_{s_1}} \sum_{\sigma, \tau \in G_{s_2}} \beta_{f,\psi,\sigma,\tau} \overline{\sigma f \tau}(q_\Theta(\psi))$$

where only finitely many of the $\beta_{f,\psi,\sigma,\tau} \in R$ are nonzero. Thus, any $w \in W_{s_2}$ has the form

$$w = \sum_{s_1 \in B} \sum_{f: s_1 \to s_2} \sum_{\psi \in \Psi'_\Phi \cap (\Psi_\Phi)_{s_1}} \alpha_{f,\psi} \bar{f}(q_\Phi(\psi)) + \sum_{s_1 \in B_\Theta} \sum_{f: s_1 \to s_2} \sum_{\psi \in \Psi'_\Theta \cap (\Psi_\Theta)_{s_1}} \sum_{\sigma, \tau \in G_{s_2}} \beta_{f,\psi,\sigma,\tau} \overline{\sigma f \tau}(q_\Theta(\psi))$$

for some coefficients $\alpha_{f,\psi}, \beta_{f,\psi,\sigma,\tau} \in R$.

Thus, the union of the finite sets of elements $q(\Psi'_\Phi)$ and $q(\Psi'_\Theta)$ suffice to generate all of $\mathbf{W}$. $\square$

This theorem and its proof are analogous to the result in [7] that submodules of finitely generated FI-modules are finitely generated. While there are several differences between that result and this one, note that a major one is that we needed to assume that the augmentation module $\mathbf{\Theta W}$ was finitely generated. Since the only theorem we have for our general formulation requires us to not only know that the ambient $\mathbf{G}$-bimodule $\mathbf{V}$ by also the augmentation module is finitely generated, we would like to be able to determine which augmentation modules are finitely generated. At the moment we have one such result, Proposition 3, which says that the augmentation ideal of an NFG synergy is finitely generated.

2.4 The symmetric synergy

While we have departed quite a bit from the paradigm of a sequence of $\mathbf{\Sigma}_n$ actions, we close out this chapter by giving a different Noetherianess result for $\mathbf{\Sigma}$-bimodules. None of our previous results depended on understanding the representation theory of the component groups $\mathbf{G}_s$, but here we will make use of the well-known representation theory of the symmetric groups in a different way than it's used in FI-module theory.

Here we denote by 1 the free $\mathbf{N}$-set with a generating set Ξ which consists of a single generator $\xi \in 1_1$.

Theorem 2. *Sub-$\mathbf{\Sigma}$-bimodules of $\mathbb{C}\mathbf{\Sigma}[1]$ are finitely generated.*

Proof. Let $\mathbf{W} \leq \mathbb{C}\mathbf{\Sigma}[1]$. Since $(\mathbb{C}\mathbf{\Sigma}[1])_n$ only has a single generator, say ξ with a slight abuse of notation, such that for any $\sigma, \tau \in \Sigma_n$ we have $\bar{\sigma}\xi\bar{\tau} = \sigma\tau\xi$ each of the $(\mathbb{C}\mathbf{\Sigma}[1])_n$ are effectively the complex group algebra $\mathbb{C}\mathbf{\Sigma}_n$.

Consider the $\mathbb{N}$-graded poset $\mathbf{P}$ where P_n is the set of isomorphism classes of irreducible complex representations of $\mathbf{\Sigma}_n$ and where $A \leq B$ in $\mathbf{P}$ for $A \in P_m$ and $B \in P_n$ when B is a summand of $\mathrm{Ind}_{\mathbf{\Sigma}_m}^{\mathbf{\Sigma}_n}(A)$. (Note that by Frobenius Reciprocity this is equivalent to A being a summand of $\mathrm{Res}_{\mathbf{\Sigma}_m}^{\mathbf{\Sigma}_n}(B)$.)

By the two-sided action of $\mathbf{\Sigma}_n$ on $\mathbb{C}\mathbf{\Sigma}_n$ it follows that sub-$\mathbf{\Sigma}$-bimodules of $\mathbb{C}\mathbf{\Sigma}[1]$ are in bijective correspondence with upsets of $\mathbf{P}$. Moreover, the size of a minimal generating set for some $\mathbf{W} \leq \mathbb{C}\mathbf{\Sigma}[1]$ is the number of minimal elements of the corresponding upset of $\mathbf{P}$. In order to show that each sub-$\mathbf{\Sigma}$-bimodule is finitely generated it then suffices to show that $\mathbf{P}$ is partially well-ordered. That is, we would like to show that any upset $U \subset P$ has only finitely many minimal elements.

Recall that isomorphism classes irreducible complex representations of $\mathbf{\Sigma}_n$ are in bijective correspondence with Young diagrams for partitions of n and that Pieri's formula indicates that the partial order of $\mathbf{P}$ is the transitive closure of the covering relation where $A \prec B$ for $A \in P_n$ and $B \in P_{n+1}$ when the Young diagram for B may be obtained from the Young diagram for A by adding a single cell.

Suppose that $\mathbf{W}$ corresponds to an upset $U \subset P$ which contains the partition $a \coloneqq (a_1, \ldots, a_k)$. Let V be the complement in P of the upset generated by a. That is,

$$V \coloneqq \{\, b \in P \mid b \not\leq a \,\}.$$

It follows that

$$V = V_1 \cup V_2 \cup \cdots \cup V_k$$

where

$$V_i \coloneqq \{\, b \in P \mid (\forall i \leq j \leq k)(b_j < a_j) \text{ and } (\forall j < i)(b_j \geq a_j) \,\}.$$

If we can show that each of the finitely many V_i have only finitely many minimal elements we will be done.

Fix some i and consider that each member $b \in V_i$ can be identified by a tuple $(c_1, \ldots, c_{i-1})$ such that $c_j < a_j$ when $i \leq j \leq k$ and $c_j = a_j$ when $i < j$ along with a tuple $(d_1, \ldots, d_{b_i}, e_1, \ldots, e_{i-1})$

such that

$$b = (c_1 + e_1, c_2 + e_2, \dots, c_{i-1} + e_{i-1}, u_1, \dots, u_\ell)$$

where u_r is the number of i for which $d_i \geq r$.

Since there are only finitely many possible tuples $(c_1, \dots, c_{i-1})$ corresponding to a member $b \in V_i$ in this manner, it suffices to show that for a fixed such $c \coloneqq (c_1, \dots, c_{i-1})$ the set S_c of all $b \in V_i$ containing it has only finitely many minimal elements under the partial order of $\mathbf{P}$. Given $b, b' \in S_c$ corresponding to $\alpha \coloneqq (d_1, \dots, d_{b_i}, e_1, \dots, e_{i-1})$ and $\alpha' \coloneqq (d'_1, \dots, d'_{b_i}, e'_1, \dots, e'_{i-1})$ we have that $b \leq b'$ in $\mathbf{P}$ if and only if $\alpha \leq \alpha'$ in the product order on $\mathbf{N}^{b_i+i-1}$. By Dickson's Lemma[13] we have that $\mathbf{N}^{b_i+i-1}$ is partially well-ordered so there are only finitely many minimal members of each S_c. $\square$

Chapter 3

Isomorphism invariant polynomials

In a slight departure from our earlier results on synergy bimodules, we now consider only a single one-sided action of the symmetric group at a time. Recall that given a set of variables X the symmetric group $\mathbf{\Sigma}_X$ of permutations of X acts on the corresponding polynomial algebra $\mathbf{R}[X]$ for some unital commutative ring $\mathbf{R}$. The polynomials invariant under this action are the *symmetric polynomials*, which themselves form an $\mathbf{R}$-algebra. A classical result of Hilbert is that certain very simple *elementary symmetric polynomials* generate this algebra of all symmetric polynomials[14, p.191].

Our Theorem 3 is a direct generalization of this to the setting of finite structures and tells us that the fundamental invariants for any collection of finite structures on a fixed finite set are the counts of embeddings of a fixed substructure.

As mentioned in the introduction, Bourbaki's original treatment of structures was fairly involved and presumably a categorification might be even more convoluted. Thus, we postpone this formal development until nd present three possible ways for the reader to think about structures in this chapter.

The first is that one might consider a finite structure as one would in model theory. That is, a *finite structure* is a pair $\mathbf{A} \coloneqq (A, \{f_i\}_{i \in I})$ where A is a finite set and the f_i form an I-indexed sequence of relations $f_i \subset A^{\rho(i)}$ where the function $\rho\colon I \to \mathbb{N}$ is the *signature* of $\mathbf{A}$. We denote by $\mathbf{Struct}^{\rho}$ the evident category and by $\mathrm{Struct}^{\rho}_{A}$ the collection of all structures of the same signature on the set A, which we call a *kinship class*. The class Struct^{ρ} of all structures with signature ρ is likewise called a *similarity class*. We will always take the index set I to be finite here.

The second way of thinking about finite structures is in the sense of Bourbaki. That is, do the same as in the preceding paragraph but allow yourself to think of a *relation* as allowing powerset operators and Cartesian products in arbitrary finite compositions. This makes it easier to see how finite topological spaces could be counted among finite structures.

The third way, which should probably be postponed on first reading, is to instead go through irst to see structures (not necessarily finite, or even built from sets at all) in their full, formal generality. This will make the following sections much more rigorous at the expense of adding extra bookkeeping while first gaining intuition.

3.1 Substructures

There is a natural categorical definition of a substructure.

Definition 27 (Substructure)**.** Given a structure $\mathbf{A}$ of signature ρ we refer to a subobject of $\mathbf{A}$ in $\mathbf{Struct}^\rho$ as a *substructure* of $\mathbf{A}$.

In the case of **Set**-structures we can give a concrete description of the poset of substructures. Given a **Set**-structure $\mathbf{A} := (A, F)$ of signature ρ we have that $\mathbf{A}$ consists of, for each $N \in \mathrm{Ob}(\mathscr{I})$, a subset $F(N) \subset \rho_A(N)$, and, for each $\nu \in \mathrm{Mor}(\mathscr{I})$ with domain N, a restriction (on both the domain and codomain) $F(\nu) = (\rho_A(\nu))|_{F(N)}$. This means that for a collection of subsets of the $\rho_A(N)$ to form a structure of signature ρ it is necessary and sufficient that given a morphism $\nu\colon N_1 \to N_2$ from $\mathscr{I}$ we have that the image of $F(N_1)$ under $\rho_A(\nu)$ is contained in $F(N_2)$.

Note that a **Set**-structure with universe A is a substructure of the structure $(A, \mathrm{id}_{\rho_A})$. A substructure of some $\mathbf{A}_1 := (A, F_1) \in \mathrm{Struct}_A^\rho$ is then another structure $\mathbf{A}_2 := (A, F_2) \in \mathrm{Struct}_A^\rho$ such that $\mathbf{A}_2 \leq \mathbf{A}_1$ in the substructure poset of $(A, \mathrm{id}_{\rho_A})$, which is equivalent to having for each $N \in \mathrm{Ob}(\mathscr{I})$ that $F_2(N) \subset F_1(N)$.

One can verify that the substructure poset of $(A, \mathrm{id}_{\rho_A})$ forms a complete lattice and hence the substructure poset $\mathbf{Sub}(\mathbf{A})$ of any $\mathbf{A} \in \mathrm{Struct}_A^\rho$ is also a complete lattice. Since the substructure poset of $(A, (\mathrm{id}_{\rho_A}))$ is a sublattice of a Boolean lattice all of these lattices are also distributive.

3.2 Finite structures

We give formal definitions of finite structures here, but if you're thinking of model-theoretic or Bourbakian structures you can ignore these definitions in favor of the ones you already have in mind.

Definition 28 (Finite signature)**.** We say that a signature $\rho\colon \mathscr{I} \to \mathbf{Fun}(\mathbf{Set}, \mathbf{Set})$ is *finite* when $\mathscr{I}$ has finitely many objects and finitely many morphisms and for each $N \in \mathrm{Ob}(\mathscr{I})$ and each finite set A we have that $\rho_A(N)$ is finite.

Definition 29 (Finite structure)**.** We say that a structure of finite signature ρ on a finite set is a *finite structure.*

Definition 30 (Finite kinship class)**.** When ρ is a finite signature and A is a finite set we say that $\mathrm{Struct}^{\rho}_{A}$ is a *finite kinship class.*

Note that each of the members of a finite kinship class are finite structures and that the kinship class itself is a finite set.

3.3 Symmetric polynomials

We consider polynomial algebras associated to finite kinship classes.

Definition 31 (Variables X^{ρ}_{A})**.** Given a finite signature ρ on an index category $\mathscr{I}$ and a finite set A we define

$$X^{\rho}_{A} \coloneqq \bigcup_{N \in \mathrm{Ob}(\mathscr{I})} \{\, x_{N,a} \mid a \in \rho_A(N) \,\} .$$

Given a commutative ring $\mathbf{R}$ and a set X we write $\mathbf{R}[X]$ to denote the free commutative unital $\mathbf{R}$-algebra generated by X and $R[X]$ to denote that algebra's universe.

Definition 32 (Monomial $y_{\mathbf{A}}$)**.** Given a finite signature ρ on an index category $\mathscr{I}$, a finite set A, and a structure $\mathbf{A} \coloneqq (A, F) \in \mathrm{Struct}^{\rho}_{A}$ we define

$$y_{\mathbf{A}} \coloneqq \prod_{N \in \mathrm{Ob}(\mathscr{I})} \prod_{a \in F(N)} x_{N,a}.$$

Note that there is always an empty structure of a given signature and hence one of the $y_{\mathbf{A}}$ will always be 1.

Definition 33 (Monomials Y_A^ρ)**.** Given a finite signature ρ on an index category $\mathscr{I}$ and a finite set A we define

$$Y_A^\rho \coloneqq \{\, y_{\mathbf{A}} \mid \mathbf{A} \in \mathrm{Struct}_A^\rho \,\}\,.$$

Definition 34 ((ρ, A) polynomial algebra)**.** Given a commutative ring $\mathbf{R}$, a finite signature ρ, and a finite set A we define the (ρ, A) *polynomial algebra* over $\mathbf{R}$ to be the subalgebra of $\mathbf{R}[X_A^\rho]$ which is generated by Y_A^ρ. We denote this algebra by $\mathbf{Pol}_A^\rho(\mathbf{R})$ and its universe by $\mathrm{Pol}_A^\rho(\mathbf{R})$.

We omit the ring $\mathbf{R}$ when we take $\mathbf{R}$ to be $\mathbb{Z}$. For example, we write $\mathbf{Pol}_A^\rho$ to indicate $\mathbf{Pol}_A^\rho(\mathbb{Z})$.

By our previous comment that $1 \in Y_A^\rho$ polynomials in $\mathrm{Pol}_A^\rho(\mathbf{R})$ can have any nonzero constant term.

In order to prove the main result of this section we will need the following lemma on the factorization of monomials in $\mathbf{Pol}_A^\rho(\mathbf{R})$.

Lemma 1. *Given $y_{\mathbf{A}_1}, \ldots, y_{\mathbf{A}_k} \in Y_A^\rho$ we have that*

$$\prod_{i=1}^{k} y_{\mathbf{A}_i} = y_{\bigvee_{i=1}^{k} \mathbf{A}_i} \mu$$

where $\mu \in \mathrm{Pol}_A^\rho$.

Proof. We induct on the number of factors k. When $k = 1$ we can take $\mu = 1$ and when $k = 2$ we can take $\mu = y_{\mathbf{A}_1 \wedge \mathbf{A}_2}$. Take $k \geq 3$ and suppose that we have the result for all $k' < k$. In this case we observe that

$$\begin{aligned}
\left(\prod_{i=1}^{k-1} y_{\mathbf{A}_i}\right) y_{\mathbf{A}_k} &= \left(y_{\bigvee_{i=1}^{k-1} \mathbf{A}_i} \mu\right) y_{\mathbf{A}_k} \\
&= \left(y_{\bigvee_{i=1}^{k-1} \mathbf{A}_i} y_{\mathbf{A}_k}\right) \mu \\
&= \left(y_{(\bigvee_{i=1}^{k-1} \mathbf{A}_i) \vee \mathbf{A}_k} \mu'\right) \mu \\
&= y_{\bigvee_{i=1}^{k} \mathbf{A}_i} \mu\mu'.
\end{aligned}$$

Since $\mu, \mu' \in \mathrm{Pol}_A^\rho$ we have that $\mu\mu' \in \mathrm{Pol}_A^\rho$, as well. $\square$

We have a natural action of $\mathbf{\Sigma}_A$ on $\mathbf{R}[X_A^\rho]$.

Definition 35 (Action υ)**.** We define a group action $\upsilon\colon \mathbf{\Sigma}_A \to \mathbf{Aut}(\mathbf{R}[X_A^\rho])$ by setting $(\upsilon(\sigma))(x_{N,a}) \coloneqq x_{N,(\rho_\sigma(N))(a)}$ and extending.

Definition 36 (Symmetric polynomial)**.** A polynomial $p \in \mathrm{Pol}^{\rho}_{A}(\mathbf{R})$ is called *symmetric* when for every $\sigma \in \Sigma_A$ we have that $(\upsilon(\sigma))(p) = p$.

Definition 37 ((ρ, A) symmetric polynomial algebra)**.** We denote by $\mathrm{SymPol}^{\rho}_{A}(\mathbf{R})$ the set of symmetric polynomials in $\mathrm{Pol}^{\rho}_{A}(\mathbf{R})$ and we denote by $\mathbf{SymPol}^{\rho}_{A}(\mathbf{R})$ the subalgebra of $\mathbf{Pol}^{\rho}_{A}(\mathbf{R})$ with universe $\mathrm{SymPol}^{\rho}_{A}(\mathbf{R})$, which we refer to as the (ρ, A) *symmetric polynomial algebra* over $\mathbf{R}$.

Of particular interest are a family of polynomials arising from the isomorphism classes of members of $\mathrm{Struct}^{\rho}_{A}$.

Definition 38 (Action ζ)**.** We define a group action $\zeta \colon \mathbf{\Sigma}_A \to \mathbf{\Sigma}_{\mathrm{Struct}^{\rho}_{A}}$ by

$$(\zeta(\sigma))(A, F) \coloneqq (A, \rho_\sigma \circ F).$$

This action is well-defined as a change in representative F doesn't change the equivalence class of monomorphisms to which $\rho_\sigma \circ F$ belongs.

Definition 39 (Isomorphism classes of structures)**.** We define

$$\mathrm{IsoStr}^{\rho}_{A} \coloneqq \{\, \mathrm{Orb}_{\zeta}(\mathbf{A}) \mid \mathbf{A} \in \mathrm{Struct}^{\rho}_{A} \,\}.$$

Definition 40 (Elementary symmetric polynomial)**.** Given a finite signature ρ, a finite set A, and an isomorphism class $\psi \in \mathrm{IsoStr}^{\rho}_{A}$ we define the *elementary symmetric polynomial* of ψ to be

$$s_\psi \coloneqq \sum_{\mathbf{A} \in \psi} y_{\mathbf{A}}.$$

Definition 41 (Polynomials S^{ρ}_{A})**.** Given a finite signature ρ and a finite set A we define

$$S^{\rho}_{A} \coloneqq \{\, s_\psi \mid \psi \in \mathrm{IsoStr}^{\rho}_{A} \,\}.$$

Proposition 4. *The elementary symmetric polynomials are symmetric polynomials.*

Proof. Let s_ψ be an elementary symmetric polynomial over $\mathbf{R}$. Since s_ψ is a sum of monomials belonging to Y_A^ρ we have that $s_\psi \in \mathrm{Pol}_A^\rho(\mathbf{R})$. Take $\sigma \in \Sigma_A$. We have that

$$\begin{aligned}
(\upsilon(\sigma))(s_\psi) &= (\upsilon(\sigma))\left(\sum_{(A,F)\in\psi}\prod_{N\in\mathrm{Ob}(\mathscr{I})}\prod_{a\in F(N)} x_{N,a}\right)\\
&= \sum_{(A,F)\in\psi}\prod_{N\in\mathrm{Ob}(\mathscr{I})}\prod_{a\in F(N)} (\upsilon(\sigma))(x_{N,a})\\
&= \sum_{(A,F)\in\psi}\prod_{N\in\mathrm{Ob}(\mathscr{I})}\prod_{a\in F(N)} x_{N,(\rho_\sigma(N))(a)}\\
&= \sum_{\substack{(\zeta(\sigma))(A,F)\\(A,F)\in\psi}}\prod_{N\in\mathrm{Ob}(\mathscr{I})}\prod_{a\in(\rho_\sigma\circ F)(N)} x_{N,a}\\
&= \sum_{(A,F)\in\psi}\prod_{N\in\mathrm{Ob}(\mathscr{I})}\prod_{a\in F(N)} x_{N,a}\\
&= s_\psi,
\end{aligned}$$

as claimed. □

Definition 42 (Magnitude of a structure)**.** Given a finite structure $\mathbf{A} \coloneqq (A, F) \in \mathrm{Struct}_A^\rho$ we define the *magnitude* of $\mathbf{A}$ to be

$$\|\mathbf{A}\| \coloneqq \sum_{N\in\mathrm{Ob}(\mathscr{I})} |F(N)|\,.$$

Definition 43 (Magnitude of an isomorphism class)**.** Given $\psi \in \mathrm{IsoStr}_A^\rho$ we define the *magnitude* of ψ to be $\|\psi\| \coloneqq \|\mathbf{A}\|$ for any $\mathbf{A} \in \psi$.

Since isomorphic structures have the same magnitude $\|\psi\|$ is well-defined.

Proposition 5. *We have that s_ψ is homogeneous of degree $\|\psi\|$.*

Proof. Observe that s_ψ is a sum of monomials, one for each member $\mathbf{A}$ of ψ. Each of these monomials have degree $\|\mathbf{A}\| = \|\psi\|$. □

Definition 44 (Variables Z_A^ρ)**.** Given a finite signature ρ on an index category $\mathscr{I}$ and a finite set A we define

$$Z_A^\rho \coloneqq \{\, z_\psi \mid \psi \in \mathrm{IsoStr}_A^\rho \,\}\,.$$

Definition 45 (Weight of a monomial)**.** The *weight* of a monomial $\prod_\psi z_\psi^{d_\psi}$ in $R[Z_A^\rho]$ is defined to be $\sum_\psi \|\psi\|\, d_\psi$.

Definition 46 (Weight of a polynomial)**.** The weight of a polynomial $p \in R[Z_A^\rho]$ is the maximum of the weights of the monomials appearing in p.

We generalize a statement of Hilbert by showing that the elementary symmetric polynomials generate the algebra of symmetric polynomials. We follow Lang's treatment[14, p.191].

Theorem 3. *Given a polynomial $f \in \mathrm{SymPol}^{\rho}_{A}(\mathbf{R})$ of degree d there exists a polynomial $g \in R[Z^{\rho}_{A}]$ of weight at most d such that $f = g|_{Z^{\rho}_{A}=S^{\rho}_{A}}$.*

Proof. Define $n \coloneqq |A|$. We induct on n. When $n = 0$ we have that $\mathbf{\Sigma}_A$ is trivial and hence each class in $\mathrm{IsoStr}^{\rho}_{A}$ contains a unique member. It follows that $\mathrm{SymPol}^{\rho}_{A}(\mathbf{R}) = \mathrm{Pol}^{\rho}_{A}(\mathbf{R})$ and the Y^{ρ}_{A} are precisely the elementary symmetric polynomials. The polynomial g can be obtained from f by replacing each occurrence of a monomial $y_{\mathbf{A}}$ in a term of f with the corresponding singleton orbit variable $z_{\{\mathbf{A}\}}$. By definition of the weight of a polynomial this choice of g will have weight precisely d.

Suppose that $n > 0$ and that we have the result for $n-1$. We induct on d. Put a total order on A so that $A = \{a_1, \ldots, a_n\}$. Define $B \coloneqq A \setminus \{a_n\}$ and define $\iota\colon B \to A$ to be inclusion given by $\iota(a_i) \coloneqq a_i$. For each $N \in \mathrm{Ob}(\mathscr{I})$ this map induces an inclusion $\rho_\iota(N)\colon \rho_B(N) \to \rho_A(N)$. Define

$$A_n \coloneqq \bigcup_{N \in \mathrm{Ob}(\mathscr{I})} \{\, x_{N,a} \mid a \in \rho_A(N) \setminus \mathrm{Im}(\rho_\iota(N)) \,\}$$

to be the collection of variables in X^{ρ}_{A} depending on a_n. We have that $f|_{A_n=0} \in \mathrm{SymPol}^{\rho}_{B}(\mathbf{R})$ so there exists some $g_1 \in R[Z^{\rho}_{B}]$ of weight at most d such that $f|_{A_n=0} = g_1|_{Z^{\rho}_{B}=S^{\rho}_{B}}$. Note that for each $s_\psi \in S^{\rho}_{B}$ there is a unique member $s'_\psi \in S^{\rho}_{A}$ such that $s_\psi = (s'_\psi)|_{A_n=0}$. By the inclusion induced by ι identify g_1 with a polynomial, which we will also call g_1, belonging to $R[Z^{\rho}_{A}]$. We find that $f|_{A_n=0} = (g_1|_{Z^{\rho}_{A}=S^{\rho}_{A}})|_{A_n=0}$. Define $f_1 \coloneqq f - g_1|_{Z^{\rho}_{A}=S^{\rho}_{A}}$. Observe that f_1 has degree at most d and is symmetric.

By applying our lemma on the factorization of monomials we can write f_1 uniquely as

$$f_1 = \sum_{\mathbf{A} \in \mathrm{Struct}^{\rho}_{A}} y_{\mathbf{A}} p_{\mathbf{A}}$$

where each $y_{\mathbf{A}}$ is the monomial factor guaranteed by that lemma. Since f_1 has no constant term each $p_{\mathbf{A}} \in \mathrm{Pol}^{\rho}_{A}(\mathbf{A})$ has degree strictly less than d. Since f_1 is symmetric the application of some $\sigma \in \Sigma_A$ would appear to give us a different such expression for f_1. It follows that if $\mathbf{A}_1 \cong \mathbf{A}_2$ then $p_{\mathbf{A}_1} = p_{\mathbf{A}_2}$. We can then collect terms to obtain $f_1 = \sum_{\psi \in \mathrm{IsoStr}^{\rho}_{A}} s_\psi p_\psi$ where $p_\psi = p_{\mathbf{A}}$ for any $\mathbf{A} \in \psi$. Applying the inductive hypothesis to the $p_{\mathbf{A}}$ we obtain $p_{\mathbf{A}} = (g_{\mathbf{A}})|_{Z^{\rho}_{A}=S^{\rho}_{A}}$ where each $g_{\mathbf{A}}$

has weight at most $d - \|\psi\|$. It follows that f_1 can be written as a polynomial g_2 in the elementary symmetric polynomials of weight at most d. That is, there exists some $g_2 \in R[Z^\rho_A]$ of weight at most d such that $f_1 = g_2|_{Z^\rho_A = S^\rho_A}$. This implies that $f = (g_1 + g_2)|_{Z^\rho_A = S^\rho_A}$ where $g_1 + g_2 \in R[Z^\rho_A]$ has weight at most d. $\square$

Although it happens that $(f_1)|_{A_n=0}= 0$ and hence each term in f_1 is divisible by some element of A_n we did not need to use this fact. We can make an observation analogous to the one in the proof in Lang (loc. cit.) in this direction, which is that by symmetry each term in f_1 must be divisible by some element of A_i for each i. This suffices in that special case because there is a minimal monomial with this property.

Most of our definitions and arguments go through if we use a suitably finite signature $\rho\colon \mathscr{I} \to \mathbf{Fun}(\mathbf{Set}, \mathbf{Set}^{\mathrm{op}})$ instead. If a similar result for this class of structures is to be proved then we must make a change at the point where we take the induced map $\rho_\iota(N)$, for this will now give us a morphism in $\mathbf{Set}^{\mathrm{op}}$ whose corresponding map in $\mathbf{Set}$ is one taking members of $\rho_A(N)$ to members of $\rho_B(N)$.

It is not the case in general that the symmetric polynomials S^ρ_A generate $\mathbf{SymPol}^\rho_A(\mathbf{R})$ freely. We give a specific example of a nontrivial algebraic relation between elementary symmetric polynomials in the next section.

3.4 Example: domain digraphs

Most of the categories of structures with which we are already acquainted have no nontrivial relators. We consider structures with a relator which is not an identity or isomorphism in order to get a flavor of the general case.

Definition 47 (Domain digraph)**.** A *domain digraph* with universe A consists of some $E \subset A^2$ and some $W \subset A$ such that for each $(a_0, a_1) \in E$ we have that $\pi(a_0, a_1) = a_0 \in W$.

We can visualize this as a digraph E on a set of vertices A where a subset $W \subset A$ of *domain vertices* are marked. Each edge in E must have its source vertex in W, although in general a domain vertex need not be the source of any edge in E. We will denote a domain digraph $\mathbf{A}$ with universe A, edge set E, and domain vertex set W by $\mathbf{A} \coloneqq (A, E, W)$.

Domain digraphs can be construed as structures in our formal sense where there are two basic relations and a morphism between them in the index category $\mathscr{I}$.

We give an example where the elementary symmetric polynomials S_A^ρ are algebraically dependent. Take $A \coloneqq \{x_0, x_1\}$. In this case we have that $A^2 = \{x_{00}, x_{01}, x_{10}, x_{11}\}$. Observe that

$$X_A^\rho = \{x_0, x_1, x_{00}, x_{01}, x_{10}, x_{11}\}$$

and

$$\begin{aligned} Y_A^\rho = \{&1, x_0, x_1, x_0x_1, \\ &x_{00}x_0, x_{00}x_0x_1, x_{01}x_0, x_{01}x_0x_1, x_{10}x_1, x_{10}x_0x_1, x_{11}x_1, x_{11}x_0x_1, \\ &x_{00}x_{01}x_0, x_{00}x_{01}x_0x_1, x_{00}x_{10}x_0x_1, x_{00}x_{11}x_0x_1, x_{01}x_{10}x_0x_1, x_{01}x_{11}x_0x_1, x_{10}x_{11}x_1, x_{10}x_{11}x_0x_1, \\ &x_{00}x_{01}x_{10}x_0x_1, x_{00}x_{01}x_{11}x_0x_1, x_{00}x_{10}x_{11}x_0x_1, x_{01}x_{10}x_{11}x_0x_1, \\ &x_{00}x_{01}x_{10}x_{11}x_0x_1\}. \end{aligned}$$

We find that

$$S_A^\rho = \{1, s_0, s_{0,1}, s_{00,0}, s_{00,0,1}, s_{01,0}, \dots\}.$$

One example of an algebraic dependence between the elementary symmetric polynomials is

$$\begin{aligned} s_{00,0}s_{01,0} &= (x_{00}x_0 + x_{11}x_1)(x_{01}x_0 + x_{10}x_1) \\ &= x_{00}x_{01}x_0^2 + x_{00}x_{10}x_0x_1 + x_{01}x_{11}x_0x_1 + x_{10}x_{11}x_1^2 \\ &= (x_{00}x_{01}x_0 + x_{10}x_{11}x_1)(x_0 + x_1) - (x_{00}x_{01}x_0x_1 + x_{10}x_{11}x_0x_1) + (x_{00}x_{10}x_0x_1 + x_{01}x_{11}x_0x_1) \\ &= s_{00,01,0}s_0 - s_{00,01,0,1} + s_{00,10,0,1}. \end{aligned}$$

More succinctly, we have

$$s_{00,0}s_{01,0} - s_{00,01,0}s_0 + s_{00,01,0,1} - s_{00,10,0,1} = 0.$$

Chapter 4

Conclusion

Our guiding light has illuminated the beginning of two paths, but we will stop just short of showing where they converge. We have seen in Proposition 3 and Theorem 2 that we can show quite flexible systems of compatible two-sided group actions have rich examples of finitely generated bimodules. A next step would be to consider how finite generation behaves under tensor products of synergy bimodules and to better understand the sizes of the relevant finite generating sets. This would exactly parallel the applications of FI-modules and would allow for proofs of similar polynomial counts.

This is in contrast to the situation of Theorem 3, where we know that polynomials in terms of numbers of embeddings of substructures suffice to compute all isomorphism-invariant properties, but only know this for finite structures on a fixed finite set. We have not here considered the colimit algebra of symmetric functions which would be a natural analogue of the usual ring of symmetric functions. More in line with our philosophy however would be to consider not a single finite structure at a time, but a family of finite structures indexed by a category. That is, we should really look at functors $\mathbf{A}\colon \mathbf{S} \to \mathbf{Struct}^{\rho}$. If we have a synergy $\mathbf{G}$ of shape $\mathbf{S}$ which acts compatibly on the $\mathbf{A}_s$ by automorphisms then we are in a generalization of the setting of FI-graphs considered in [15].

Given how many natural families of symmetric group actions are finitely generated, it is plausible that our Theorem 3 is the most basic version of a result which is more properly about finite generation of a $\mathbf{\Sigma}$-bialgebra of symmetric polynomials. The advantage of such a result is that for a given class of finite structures, such as graphs or simplicial complexes, we may be able to determine a systematic way of counting small substructures which can test any given first-order property for structures on an arbitrary finite set of elements. This would in turn give a class of algorithms for testing finite structures for a wide range of properties.

Numerous existing results may already fall under this general paradigm without having been realized as special cases of a the same phenomenon. For just a single example, in the theory of quadratic spaces over finite fields it is known that the number of copies of each isomorphism class of line contained in a given quadratic space is a complete isometric invariant. This seems at odds with the plausible complexity with which the same multisets of lines might be able to be arranged in a high enough dimensional space over a finite field. It may then be that the isomorphism problem for quadratic spaces over finite fields has low «logical weight» in that the elementary symmetric polynomials appearing in the most efficient encoding of it only need to count the small (i.e. 1-dimensional) substructures.

In a different direction, our Theorem 1 didn't rely on two-sided actions but did make use of a tool which did not appear elsewhere: the module of coinvariants $\mathbf{\Phi V}$ for a $\mathbf{G}$-bimodule $\mathbf{V}$. Even more to the point, we can consider the coinvariants module $\mathbf{\Phi G}[\Psi]$ for a regular $\mathbf{G}$-bimodule $\mathbf{RG}[\Psi]$. While there is some room in deciding what the best choice of generators Ψ is here, $\mathbf{\Phi G}[\Psi]$ looks quite close to what should properly be called $H_1(\mathbf{G};\mathbf{R})$. That is, if a synergy $\mathbf{G}$ plays the role of a single group in our generalized FI-module theory and an object very similar to the first homology group appears at a critical place in our argument, it stands to reason that there should be a (co)homology theory for synergies. Given the wild success of group cohomology in unifying seemingly unrelated notions in group theory, combinatorics, and topology, it's fair to think that a viable theory of synergy cohomology would be of interest.

Another important condition in Theorem 1 was that the ring $\mathbf{R}\{\Xi\}$ was Noetherian. The main way we can actually establish this in examples is by assuming that $\mathbf{R}$ itself is Noetherian and then showing that Ξ is a finite generating set for $\mathbf{Esc}(\mathbf{S})$. The fact that the finite generation of the escalation monoid was trivial in the original proof for FI-modules but generalized to a discussion which contains finite generation of isotone maps for posets as well as inner automorphisms of groups as special cases and is strongly reminiscent of the automorphisms in [10] feels too natural to be a mathematical coincidence, if such things even truly exist.

For those who have already read the appendix we have some final comments about structures as they are presented here. In Proposition 12 we show that a wide class of categories of structures built from sets can be embedded into categories of structures in the sense of model theory. That is, we can view any such category $\mathbf{Struct}^{\rho}$ as actually sitting inside of a category $\mathbf{Struct}^{\rho^{\mathscr{C}}}$ whose objects are sets equipped with indexed classes of basic relations which are bona fide n-ary relations on that underlying set. The upshot here is not that the study of more general kinds of structures is irrelevant. Indeed, we saw throughout the rest of this thesis that the indexing category often plays a crucial role in finite generation arguments and having a proper-class size indexing set as is in the case for the Cartesian Yoneda embedding is not very close to finiteness.

What is brought to the fore is rather that there is a tension between simpler descriptions of basic relations and a simpler indexing of those relations. Bourbaki already may have had this feeling since the introduction of the powerset operation when defining relations is a succinct way to allow topologies to be treated in the same breath as algebras or graphs. Such considerations might have the flavor of general nonsense, but as we have seen with the developments around finite generation of $\mathbf{Esc}(\mathbf{S})$ the shape of an indexing category can make the difference between a bimodule being finitely generated or not. As this can in turn control the existence of polynomial functions for the growth of combinatorial quantities we find that the manner in which our combinatorial objects are built is, after all, a salient feature, not an afterthought.

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Appendix A

Structures

A.1 Basic definitions

A.1.1 Definition of a structure

Given categories $\mathscr{C}$ and $\mathscr{D}$ we denote by $\mathrm{Fun}(\mathscr{C},\mathscr{D})$ the class of functors from $\mathscr{C}$ to $\mathscr{D}$ and we denote by $\mathbf{Fun}(\mathscr{C},\mathscr{D})$ the functor category from $\mathscr{C}$ to $\mathscr{D}$.

Definition 48 (Presignature)**.** Given an index category $\mathscr{I}$ and categories $\mathscr{C}$ and $\mathscr{D}$ we refer to a functor $\rho\colon \mathscr{I} \to \mathbf{Fun}(\mathscr{C},\mathscr{D})$ as a *presignature*.

To each presignature we associate another functor. Given a category $\mathscr{C}$ we write $\mathrm{Ob}(\mathscr{C})$ to indicate the class of objects of $\mathscr{C}$ and $\mathrm{Mor}(\mathscr{C})$ to indicate the class of morphisms of $\mathscr{C}$.

Definition 49 (Extractor)**.** Given a presignature $\rho\colon \mathscr{I} \to \mathbf{Fun}(\mathscr{C},\mathscr{D})$ the *extractor* $\rho\_\colon \mathscr{C} \to \mathbf{Fun}(\mathscr{I},\mathscr{D})$ is defined as follows. For $A \in \mathrm{Ob}(\mathscr{C})$ we define $\rho_A\colon \mathscr{I} \to \mathscr{D}$ by $\rho_A(N) \coloneqq (\rho(N))(A)$ for each $N \in \mathrm{Ob}(\mathscr{I})$ and $\rho_A(\nu) \coloneqq (\rho(\nu))_A$ for each $\nu \in \mathrm{Mor}(\mathscr{I})$. For each morphism $h\colon A \to B$ in $\mathscr{C}$ we define $\rho_h\colon \rho_A \to \rho_B$ by $(\rho_h)_N \coloneqq (\rho(N))(h)$ for each $N \in \mathrm{Ob}(\mathscr{I})$.

Proposition 6. *The extractor $\rho\_\colon \mathscr{C} \to \mathbf{Fun}(\mathscr{I},\mathscr{D})$ of a presignature $\rho\colon \mathscr{I} \to \mathbf{Fun}(\mathscr{C},\mathscr{D})$ is a functor.*

Proof. We show that $\rho\_$ takes objects to objects. Given $A \in \mathrm{Ob}(\mathscr{C})$ we show that $\rho_A\colon \mathscr{I} \to \mathscr{D}$ is a functor. Given $N \in \mathrm{Ob}(\mathscr{I})$ we have that $\rho(N)\colon \mathscr{C} \to \mathscr{D}$ is a functor and hence $\rho_A(N) = (\rho(N))(A) \in \mathrm{Ob}(\mathscr{D})$. Given $\nu \in \mathrm{Mor}(\mathscr{I})$ we have that $\rho(\nu)$ is a natural transformation and hence

$\rho_A(\nu) = (\rho(\nu))_A \in \mathrm{Mor}(\mathscr{D})$. Thus, ρ_A takes objects to objects and morphisms to morphisms. Observe that

$$\rho_A(\mathrm{id}_N) = (\rho(\mathrm{id}_N))_A = (\mathrm{id}_{\rho(N)})_A = \mathrm{id}_{(\rho(N))(A)} = \mathrm{id}_{\rho_A(N)}$$

so ρ_A takes identities to identities. Given morphisms $\nu_1\colon N_1 \to N_2$ and $\nu_2\colon N_2 \to N_3$ in $\mathscr{I}$ we have that

$$\rho_A(\nu_2 \circ \nu_1) = (\rho(\nu_2 \circ \nu_1))_A = (\rho(\nu_2) \circ \rho(\nu_1))_A = (\rho(\nu_2))_A \circ (\rho(\nu_1))_A = \rho_A(\nu_2) \circ \rho_A(\nu_1)$$

so ρ_A respects composition of morphisms. Thus, ρ_A is a functor and $\rho\_$ takes objects to objects.

We show that $\rho\_$ takes morphisms to morphisms. Given a morphism $h\colon A \to B$ in $\mathscr{C}$ we show that $\rho_h\colon \rho_A \to \rho_B$ is a natural transformation. Given a morphism $\nu\colon N_1 \to N_2$ in $\mathscr{I}$ we have that $\rho(\nu)\colon \rho(N_1) \to \rho(N_2)$ is a natural transformation so

$$\begin{aligned} \rho_B(\nu) \circ (\rho_h)_{N_1} &= (\rho(\nu))_B \circ (\rho(N_1))(h) \\ &= (\rho(N_2))(h) \circ (\rho(\nu))_A \\ &= (\rho_h)_{N_2} \circ \rho_A(\nu) \end{aligned}$$

and hence ρ_h is also a natural transformation, which is a morphism in $\mathbf{Fun}(\mathscr{I}, \mathscr{D})$.

We show that $\rho\_$ takes identities to identities. Given an object $A \in \mathrm{Ob}(\mathscr{C})$ and an object $N \in \mathrm{Ob}(\mathscr{I})$ we have that

$$(\rho_{\mathrm{id}_A})_N = (\rho(N))(\mathrm{id}_A) = \mathrm{id}_{(\rho(N))(A)} = \mathrm{id}_{\rho_A(N)}$$

so ρ_{id_A} is the identity natural transformation of ρ_A.

We show that $\rho\_$ respects composition of morphisms. Given morphisms $h_1\colon A_1 \to A_2$ and $h_2\colon A_2 \to A_3$ in $\mathscr{C}$ and $N \in \mathrm{Ob}(\mathscr{I})$ we have that

$$(\rho_{h_2 \circ h_1})_N = (\rho(N))(h_2 \circ h_1) = (\rho(N))(h_2) \circ (\rho(N))(h_1) = (\rho_{h_2})_N \circ (\rho_{h_1})_N$$

so $\rho_{h_2 \circ h_1} = \rho_{h_2} \circ \rho_{h_1}$, as desired. □

Recall the categorical formulation of images.

Definition 50 (Factorization)**.** Given a morphism $h\colon A \to B$ in a category $\mathscr{C}$ we refer to a triple (V, θ, ψ) where $V \in \mathrm{Ob}(\mathscr{C})$, $\theta\colon A \to V$, $\psi\colon V \to B$, and $h = \psi \circ \theta$ as a *factorization* of h.

Definition 51 (Image candidate)**.** Given a morphism $h\colon A \to B$ in a category $\mathscr{C}$ we refer to a factorization (V, θ, ψ) of h as an *image candidate* for h when ψ is monic.

Definition 52 (Image triple)**.** Given a morphism $h\colon A \to B$ in a category $\mathscr{C}$ we say that an image candidate (V_1, θ_1, ψ_1) is an *image triple* for h when given any image candidate (V_2, θ_2, ψ_2) for h there exists a unique morphism $s\colon V_1 \to V_2$ such that $\psi_2 \circ s = \psi_1$.

Definition 53 (Image of a morphism)**.** Given a morphism $h\colon A \to B$ in a category $\mathscr{C}$ for which an image triple (V, θ, ψ) exists the *image* $\mathrm{Im}(h)$ of h is the subobject of B containing ψ.

The image of a morphism is well-defined when it exists by the universal property of image triples.

We are interested in those presignatures which support taking images in a certain sense.

Definition 54 (Signature)**.** Given a presignature $\rho\colon \mathscr{I} \to \mathbf{Fun}(\mathscr{C}, \mathscr{D})$ we say that ρ is a $(\mathscr{C}, \mathscr{D})$-*signature* on the index category $\mathscr{I}$ when given any monomorphism $F\colon U \hookrightarrow \rho_A$ in $\mathbf{Fun}(\mathscr{I}, \mathscr{D})$ and any morphism $h\colon A \to B$ in $\mathscr{C}$ we have that $\mathrm{Im}(\rho_h \circ F)$ exists in $\mathbf{Fun}(\mathscr{I}, \mathscr{D})$. When $\mathscr{C} = \mathscr{D}$ we refer to a $(\mathscr{C}, \mathscr{D})$-signature on $\mathscr{I}$ as a $\mathscr{C}$*-signature* on $\mathscr{I}$.

Definition 55 (Source of a signature)**.** Given a signature $\rho\colon \mathscr{I} \to \mathbf{Fun}(\mathscr{C}, \mathscr{D})$ we refer to $\mathscr{C}$ as the *source* of ρ and say that ρ is a $\mathscr{C}$*-sourced* signature.

Definition 56 (Target of a signature)**.** Given a signature $\rho\colon \mathscr{I} \to \mathbf{Fun}(\mathscr{C}, \mathscr{D})$ we refer to $\mathscr{D}$ as the *target* of ρ and say that ρ is a $\mathscr{D}$*-targeted* signature.

We give some examples of signatures. We denote by $\mathscr{I}_I$ the category whose objects form the set $\{\, N_i \mid i \in I \,\}$ and whose morphisms are all identities. Given $n \in \mathbb{N}$ we define $\mathscr{I}_n \coloneqq \mathscr{I}_{\{1,\ldots,n\}}$. We write N rather than N_1 for the single object of $\mathscr{I}_1$.

We make use of the following characterization of $\mathbf{Fun}(\mathscr{I}_I, \mathscr{D})$ for any category $\mathscr{D}$.

Definition 57 (Sequence category)**.** Given a set I and a category $\mathscr{D}$ the *sequence category* $\mathscr{D}^I$ of $\mathscr{D}$ indexed by I is defined as follows. The objects of $\mathscr{D}^I$ are the I-indexed sequences $\{A_i\}_{i\in I}$ of objects of $\mathscr{D}$. A morphism from $\{A_i\}_{i\in I}$ to $\{B_i\}_{i\in I}$ is an I-indexed sequence $\{h_i\colon A_i \to B_i\}_{i\in I}$ of morphisms of $\mathscr{D}$. The identity morphism of $\{A_i\}_{i\in I}$ is $\{\mathrm{id}_{A_i}\colon A_i \to A_i\}_{i\in I}$. Composition of morphisms is performed componentwise. That is, if $h_1\colon A_1 \to A_2$ and $h_2\colon A_2 \to A_3$ are morphisms in $\mathscr{D}^I$ then we define $h_2 \circ h_1\colon A_1 \to A_3$ by $(h_2 \circ h_1)_i \coloneqq (h_2)_i \circ (h_1)_i$.

In other words, $\mathscr{D}^I$ is the I^{th} direct power of the category $\mathscr{D}$.

It is evident that $\mathbf{Fun}(\mathscr{I}_I, \mathscr{D})$ is canonically isomorphic to $\mathscr{D}^I$. Given $n \in \mathbb{N}$ we define $\mathscr{D}^n := \mathscr{D}^{\{1,\ldots,n\}}$. There is also a canonical isomorphism between $\mathscr{D}^1$ and $\mathscr{D}$ itself. Throughout we suppress these isomorphisms and speak of objects and morphisms of $\mathscr{D}^I$ rather than the corresponding functors from $\mathscr{I}_I$ to $\mathscr{D}$ and their natural transformations wherever they appear.

Observe that any construction in $\mathscr{D}^I$ is a sequence of constructions in $\mathscr{D}$. Monomorphisms in $\mathscr{D}^I$ are sequences of monomorphisms in $\mathscr{D}$, factorizations in $\mathscr{D}^I$ are sequences of factorizations in $\mathscr{D}$, subobjects in $\mathscr{D}^I$ are sequences of subobjects in $\mathscr{D}$, and so forth.

We have a convenient criterion for a presignature to be a signature on $\mathscr{I}_I$.

Proposition 7. *Suppose that $\rho\colon \mathscr{I}_I \to \mathbf{Fun}(\mathscr{C}, \mathscr{D})$ is a presignature such that $\mathscr{D}$ has all images. We have that ρ is a signature.*

Proof. Suppose that $F\colon U \hookrightarrow \rho_A$ is a monomorphism in $\mathbf{Fun}(\mathscr{I}_I, \mathscr{D})$ and that $h\colon A \to B$ is a morphism in $\mathscr{C}$. Since $\mathscr{D}$ has all images each of the components $(\rho_h \circ F)_i$ of $\rho_h \circ F$ has an image in $\mathscr{D}$ and this sequence of images is the image of $\rho_h \circ F$ in $\mathbf{Fun}(\mathscr{I}_I, \mathscr{D})$. $\square$

All of the following signatures have index category $\mathscr{I}_I$ for some I. We will see signatures with more involved index categories later. Note that $\mathbf{Set}$ and $\mathbf{Set}^{\text{op}}$ have all images.

Definition 58 (Identity signature)**.** Given a category $\mathscr{C}$ which has all images the *identity signature* on $\mathscr{C}$ is the functor $\rho\colon \mathscr{I}_1 \to \mathbf{Fun}(\mathscr{C}, \mathscr{C})$ where $\rho(N) := \mathrm{id}_{\mathscr{C}}$ where $\mathrm{id}_{\mathscr{C}}$ is the identity functor of $\mathscr{C}$.

Definition 59 (n-set functor)**.** Given $n \in \mathbb{N}$ denote by $\binom{-}{\leq n}$ the functor from $\mathbf{Set}$ to $\mathbf{Set}$ which takes a set A to the collection $\binom{A}{\leq n} := \bigcup_{i=1}^{n} \binom{A}{n}$ of nonempty subsets of size at most n in A and takes a function $h\colon A \to B$ to the induced map from $\binom{A}{\leq n}$ to $\binom{B}{\leq n}$. We refer to $\binom{-}{\leq n}$ as the *n-set functor*.

Definition 60 (n-hypergraph signature)**.** The *n-hypergraph signature* is the functor $\rho\colon \mathscr{I}_1 \to \mathbf{Fun}(\mathbf{Set}, \mathbf{Set})$ where $\rho(N) := \binom{-}{\leq n}$.

Definition 61 (n^{th} Cartesian power functor)**.** Given $n \in \mathbb{N}$ denote by $\_^n$ the functor from $\mathbf{Set}$ to $\mathbf{Set}$ which takes a set A to the collection of n-tuples A^n over A and takes a function $h\colon A \to B$ to the induced map from A^n to B^n. We refer to $\_^n$ as the *n^{th} Cartesian power functor*.

Definition 62 (Cartesian signature)**.** Given an index set I and a function $\tilde{\rho}\colon I \to \mathbb{N}$ the *Cartesian signature* of $\tilde{\rho}$ is the functor $\rho\colon \mathscr{I}_I \to \mathbf{Fun}(\mathbf{Set}, \mathbf{Set})$ given by $\rho(N_i) := \_^{\tilde{\rho}(i)}$.

Definition 63 (Powerset functor)**.** Denote by Sb the functor from **Set** to **Set** which takes a set A to the collection of subsets $\mathrm{Sb}(A)$ of A and takes a function $h\colon A \to B$ to the induced map from $\mathrm{Sb}(A)$ to $\mathrm{Sb}(B)$. We refer to Sb as the *powerset functor*.

Definition 64 (Hypergraph signature)**.** The *hypergraph signature* is the functor $\rho\colon \mathscr{I}_1 \to \mathbf{Fun}(\mathbf{Set}, \mathbf{Set})$ given by $\rho(N) \coloneqq \mathrm{Sb}$.

Definition 65 (Contravariant powerset functor)**.** Denote by $\mathrm{Sb}^{\mathrm{op}}$ the functor from **Set** to $\mathbf{Set}^{\mathrm{op}}$ which takes a set A to the collection of subsets $\mathrm{Sb}(A)$ of A and takes a function $h\colon A \to B$ to the induced map from $\mathrm{Sb}(B)$ to $\mathrm{Sb}(A)$. We refer to $\mathrm{Sb}^{\mathrm{op}}$ as the *contravariant powerset functor*.

Definition 66 (Pseudospace signature)**.** The *pseudospace signature* is the functor $\rho\colon \mathscr{I}_1 \to \mathbf{Fun}(\mathbf{Set}, \mathbf{Set}^{\mathrm{op}})$ given by $\rho(N) \coloneqq \mathrm{Sb}^{\mathrm{op}}$.

Our central objects of study are manufactured from signatures.

Definition 67 (Structure)**.** Given a $(\mathscr{C}, \mathscr{D})$-signature ρ on an index category $\mathscr{I}$ and $A \in \mathrm{Ob}(\mathscr{C})$ we refer to a subobject $\mathbf{A}$ of ρ_A in the category $\mathbf{Fun}(\mathscr{I}, \mathscr{D})$ as a $(\mathscr{C}, \mathscr{D})$*-structure* of signature ρ on A (or as a ρ*-structure* when we want to emphasize the signature). When $\mathscr{C} = \mathscr{D}$ we refer to a $(\mathscr{C}, \mathscr{D})$-structure as a $\mathscr{C}$*-structure*.

We will often indicate a structure by giving a member of the corresponding equivalence class of monomorphisms into ρ_A. That is, we will introduce a structure $\mathbf{A}$ of signature ρ by saying something like "consider a $(\mathscr{C}, \mathscr{D})$-structure $\mathbf{A}$ of signature ρ containing F" where F is a monomorphism in $\mathrm{Fun}(\mathscr{I}, \mathscr{D})$ with codomain ρ_A and $\mathbf{A}$ is understood to be the equivalence class of monomorphisms which is the corresponding subobject of ρ_A. If we want to be even more succinct we will write $\mathbf{A} \coloneqq (A, F)$ where F is a monomorphism with codomain ρ_A.

A.1.2 Parts of a structure

We name the various basic parts of a structure.

Definition 68 (Universe)**.** Given a structure $\mathbf{A}$ on an object A we refer to A as the *universe* of $\mathbf{A}$.

Definition 69 (Relation, arity of a relation)**.** Given a structure $\mathbf{A}$ on an object A of signature $\rho\colon \mathscr{I} \to \mathbf{Fun}(\mathscr{C}, \mathscr{D})$ and $N \in \mathrm{Ob}(\mathscr{I})$ we refer to the class of morphisms $\mathbf{A}_N \coloneqq \{\, F_N \mid F \in \mathbf{A} \,\}$ in $\mathscr{D}$ as the *relation* of $\mathbf{A}$ at N. We say that $\mathbf{A}_N$ has *arity* $\rho(N)$ or that $\mathbf{A}_N$ is $\rho(N)$*-ary*.

There is a corresponding idea for morphisms of $\mathscr{I}$. Given a category $\mathscr{D}$ we denote by $\mathbf{Mor}(\mathscr{D})$ the morphism category whose objects are the morphisms of $\mathscr{D}$ and whose morphisms are natural transformations between the corresponding diagrams in $\mathscr{D}$. Given a natural transformation $\eta\colon X \to Y$ of functors from $\mathscr{I}$ to $\mathscr{D}$ and a morphism $\nu\colon N_1 \to N_2$ in $\mathscr{I}$ we obtain a morphism from $X(\nu)$ to $Y(\nu)$ in $\mathbf{Mor}(\mathscr{D})$, which we refer to as the *component* η_ν of η at ν in analogy with the usual components of a natural transformation.

Definition 70 (Relator, arity of a relator)**.** Given a structure $\mathbf{A}$ on an object A of signature $\rho\colon \mathscr{I} \to \mathbf{Fun}(\mathscr{C}, \mathscr{D})$ and $\nu \in \mathrm{Mor}(\mathscr{I})$ we refer to the class of morphisms $\mathbf{A}_\nu := \{\, F_\nu \mid F \in \mathbf{A} \,\}$ in $\mathbf{Mor}(\mathscr{D})$ as the *relator* of $\mathbf{A}$ at ν. We say that $\mathbf{A}_\nu$ has *arity* $\rho(\nu)$ or that $\mathbf{A}_\nu$ is $\rho(\nu)$*-ary*.

In many contexts it will happen that $\mathbf{A}_N$ is actually a subobject of $\rho_A(N)$. Traditionally relations on a set are defined without reference to a particular structure. One possible generalization of this is to take a relation on A of arity $\rho(N)$ to be a subobject of $\rho_A(N)$, but it is not clear that $\mathbf{A}_N$ is always a subobject of $\rho_A(N)$ for structures as we have defined them. Similar comments hold for an extrinsic definition of relators.

Definition 71 (Source)**.** Given a $(\mathscr{C}, \mathscr{D})$-structure $\mathbf{A}$ we refer to $\mathscr{C}$ as the *source* of $\mathbf{A}$ and say that $\mathbf{A}$ is a $\mathscr{C}$*-sourced* structure.

Definition 72 (Target)**.** Given a $(\mathscr{C}, \mathscr{D})$-structure $\mathbf{A}$ we refer to $\mathscr{D}$ as the *target* of $\mathbf{A}$ and say that $\mathbf{A}$ is a $\mathscr{D}$*-targeted* structure.

A.1.3 Categories of structures

We consider categories whose objects are structures with a common signature.

Definition 73 (Similarity class)**.** Given a signature ρ we refer to the class of all structures of signature ρ as the ρ *similarity class*, which we denote by Struct^ρ.

Definition 74 (Similar structures)**.** We say that two structures $\mathbf{A}$ and $\mathbf{B}$ of the same signature ρ are *similar structures* or that $\mathbf{A}$ and $\mathbf{B}$ are *of the same similarity type*.

A homomorphism from a structure $\mathbf{A}$ to a structure $\mathbf{B}$ of the same similarity type ρ should be a morphism h from the universe A of $\mathbf{A}$ to the universe B of $\mathbf{B}$ which "respects the structure of $\mathbf{A}$ and $\mathbf{B}$". In order to formalize this we make use of the extractor of ρ.

Definition 75 (Image of a structure)**.** Given a $(\mathscr{C}, \mathscr{D})$-signature ρ, objects $A, B \in \mathrm{Ob}(\mathscr{C})$, a morphism $h\colon A \to B$, and a structure $\mathbf{A}$ of signature ρ on A containing F we refer to $h(\mathbf{A}) \coloneqq \mathrm{Im}(\rho_h \circ F)$ as the *image* of $\mathbf{A}$ under h.

Were we to use presignatures rather than signatures to define structures the image of $\mathbf{A}$ under $h\colon A \to B$ might not exist, in which case our lives would be much harder.

Definition 76 (Morphism of structures)**.** Let $\mathbf{A}$ be a structure on an object A of signature ρ and let $\mathbf{B}$ be a structure on an object B of signature ρ. We say that a morphism $h\colon A \to B$ is a *morphism* from $\mathbf{A}$ to $\mathbf{B}$ when $h(\mathbf{A}) \leq \mathbf{B}$ as subobjects of ρ_B. We write $h\colon \mathbf{A} \to \mathbf{B}$ to indicate that h is a morphism from $\mathbf{A}$ to $\mathbf{B}$.

Definition 77 (Category of structures of signature ρ)**.** We denote by $\mathbf{Struct}^\rho$ the *category of structures of signature* ρ (or the *category of* ρ*-structures*) whose objects are the structures of similarity type $\rho\colon \mathscr{I} \to \mathbf{Fun}(\mathscr{C}, \mathscr{D})$, whose morphisms are morphisms of structures, whose identity morphisms are those induced by the identity morphisms of $\mathscr{C}$, and whose composition of morphisms is given by composition of underlying morphisms in $\mathscr{C}$.

In order to establish that $\mathbf{Struct}^\rho$ is indeed a category we need the following lemmas.

Lemma 2 (Composition is isotone)**.** *Suppose that* $\mathscr{C}$ *is a category with* A_1*,* A_2*,* A_3*, and* A_4 *objects of* $\mathscr{C}$*,* $h_1\colon A_1 \hookrightarrow A_3$*,* $h_2\colon A_2 \hookrightarrow A_3$*, and* $h_3\colon A_3 \to A_4$ *such that* $h_1 \leq h_2$ *in which* $\mathrm{Im}(h_3 \circ h_1)$ *and* $\mathrm{Im}(h_3 \circ h_2)$ *exist. We have that* $\mathrm{Im}(h_3 \circ h_1) \leq \mathrm{Im}(h_3 \circ h_2)$*.*

Proof. Let (V_1, θ_1, ψ_1) be an image triple for $h_3 \circ h_1$ and let (V_2, θ_2, ψ_2) be an image triple for $h_3 \circ h_2$. Since $h_1 \leq h_2$ there exists a morphism $h_4\colon A_1 \to A_2$ such that $h_1 = h_2 \circ h_4$. It follows that

$$h_3 \circ h_1 = h_3 \circ h_2 \circ h_4 = \psi_2 \circ \theta_2 \circ h_4$$

so $(V_2, \theta_2 \circ h_4, \psi_2)$ is an image candidate for $h_3 \circ h_1$. Since (V_1, θ_1, ψ_1) is an image triple for $h_3 \circ h_1$ there exists a morphism $s\colon V_1 \to V_2$ such that $\psi_2 \circ s = \psi_1$. This implies that $\psi_1 \leq \psi_2$ and hence $\mathrm{Im}(h_3 \circ h_1) \leq \mathrm{Im}(h_3 \circ h_2)$. $\square$

Lemma 3 (Morphism composition)**.** *Suppose that* $\mathscr{C}$ *is a category with* $A_i \in \mathrm{Ob}(\mathscr{C})$ *for* $i \in \{1, 2, 3, 4, 5, 6\}$*,* $h_1\colon A_1 \to A_2$*,* $h_2\colon A_2 \to A_3$*,* $h_3\colon A_4 \hookrightarrow A_1$*,* $h_4\colon A_5 \hookrightarrow A_2$*, and* $h_5\colon A_6 \hookrightarrow A_3$ *such that*

$\mathrm{Im}(h_1 \circ h_3)$, $\mathrm{Im}(h_2 \circ h_4)$, *and* $\mathrm{Im}(h_2 \circ h_1 \circ h_3)$ *exist. Suppose also that* $\psi_1 \in \mathrm{Im}(h_1 \circ h_3)$, $\mathrm{Im}(h_2 \circ \psi_1)$ *exists,* $\mathrm{Im}(h_1 \circ h_3) \leq \mathrm{Im}(h_4)$, *and* $\mathrm{Im}(h_2 \circ h_4) \leq \mathrm{Im}(h_5)$. *We have that* $\mathrm{Im}(h_2 \circ h_1 \circ h_3) \leq \mathrm{Im}(h_5)$.

Proof. By the assumption that $\mathrm{Im}(h_1 \circ h_3) \leq \mathrm{Im}(h_4)$ we have that $\psi_1 \leq h_4$. Let (V_1, θ_1, ψ_1) be an image triple for $h_1 \circ h_3$, which must exist by our assumption that $\psi_1 \in \mathrm{Im}(h_1 \circ h_3)$. By Lemma 2 we have that

$$\mathrm{Im}(h_2 \circ \psi_1) \leq \mathrm{Im}(h_2 \circ h_4) \leq \mathrm{Im}(h_5).$$

Since $h_2 \circ h_1 \circ h_3 = h_2 \circ \psi_1 \circ \theta_1$ it suffices to show that $\mathrm{Im}(h_2 \circ \psi_1 \circ \theta_1) \leq \mathrm{Im}(h_2 \circ \psi_1)$.

Let (V_2, θ_2, ψ_2) be an image triple for $h_2 \circ \psi_1 \circ \theta_1$ and let (V_3, θ_3, ψ_3) be an image triple for $h_2 \circ \psi_1$. Since $h_2 \circ \psi_1 = \psi_3 \circ \theta_3$ we have that $h_2 \circ \psi_1 \circ \theta_1 = \psi_3 \circ \theta_3 \circ \psi_1$ and hence $(V_3, \theta_3 \circ \theta_1, \psi_3)$ is an image candidate for $h_2 \circ \psi_1 \circ \theta_1$. By the universal property of the image triple of $h_2 \circ \psi_1 \circ \theta_1$ we find that there exists a morphism $s\colon V_2 \to V_3$ such that $\psi_3 \circ s = \psi_2$. This implies that $\psi_2 \leq \psi_3$ and hence

$$\mathrm{Im}(h_2 \circ \psi_1 \circ \theta_1) = \mathrm{Im}(\psi_2) \leq \mathrm{Im}(\psi_3) = \mathrm{Im}(h_2 \circ \psi_1),$$

as desired. □

We can now prove that structures form categories.

Proposition 8. *We have that* $\mathbf{Struct}^\rho$ *is a category for any signature* ρ.

Proof. We show that morphisms compose. Let $\mathbf{A}_i := (A_i, F_i)$ with $F_i\colon U_i \hookrightarrow \rho_{A_i}$ for $i \in \{1, 2, 3\}$. Suppose that $h_1\colon \mathbf{A}_1 \to \mathbf{A}_2$ and $h_2\colon \mathbf{A}_2 \to \mathbf{A}_3$ are morphisms. We must establish that $h_2 \circ h_1$ is a morphism from $\mathbf{A}_1$ to $\mathbf{A}_3$, so we need that $\mathrm{Im}(h_2 \circ h_2 \circ F_1) \leq \mathbf{A}_3$. Unraveling definitions we find that this is precisely the situation in Lemma 3, so we have that $h_2 \circ h_1\colon \mathbf{A}_1 \to \mathbf{A}_3$.

That composition of morphisms in $\mathbf{Struct}^\rho$ is associative follows directly from the associativity of composition in $\mathscr{C}$. Similarly, identity morphisms in $\mathbf{Struct}^\rho$ satisfy the requisite identity because identity morphisms in $\mathscr{C}$ do so. □

Definition 78 (Kinship class)**.** Given a signature $\rho\colon \mathscr{I} \to \mathbf{Fun}(\mathscr{C}, \mathscr{D})$ and an object A of $\mathscr{C}$ we refer to the class of all structures of signature ρ with universe A as the (ρ, A) *kinship class*, which we denote by Struct^ρ_A.

Definition 79 (Kindred structures)**.** We say that two structures $\mathbf{A}$ and $\mathbf{B}$ of the same similarity type with the same universe are *kindred structures* or that $\mathbf{A}$ and $\mathbf{B}$ are *of the same kinship type.*

A.2 Examples of structures

We give some examples of categories of structures.

A.2.1 Pairs

Take $\rho\colon \mathscr{I}_1 \to \mathbf{Fun}(\mathscr{C}, \mathscr{C})$ to be the identity signature on a category $\mathscr{C}$. Recall that by definition of the identity signature $\mathscr{C}$ must have all images.

Definition 80 (Pair in $\mathscr{C}$)**.** Given a category $\mathscr{C}$ a *pair* in $\mathscr{C}$ (or a $\mathscr{C}$*-pair*) is an ordered pair $(A, \mathrm{Im}(F))$ where $\mathrm{Im}(F)$ is a subobject of A in $\mathscr{C}$.

Definition 81 (Pair class in $\mathscr{C}$)**.** We refer to the class of all pairs in $\mathscr{C}$ as the *pair class in* $\mathscr{C}$ (or the $\mathscr{C}$*-pair class*), which we denote by $\mathrm{Pair}(\mathscr{C})$.

We will usually write (A, F) rather than $(A, \mathrm{Im}(F))$ and remember that (A, F_1) and (A, F_2) are the same pair in $\mathscr{C}$ when $\mathrm{Im}(F_1) = \mathrm{Im}(F_2)$. Note that $(A_1, F_1) \neq (A_2, F_2)$ as pairs when $A_1 \neq A_2$, even if the domains of F_1 and F_2 are isomorphic.

Definition 82 (Morphism of pairs in $\mathscr{C}$)**.** Given a category $\mathscr{C}$, pairs $\mathbf{A}_1 \coloneqq (A_1, F_1)$ and $\mathbf{A}_2 \coloneqq (A_2, F_2)$, and a morphism $h\colon A_1 \to A_2$ we say that h is a *morphism* from $\mathbf{A}_1$ to $\mathbf{A}_2$ and write $h\colon \mathbf{A}_1 \to \mathbf{A}_2$ when $\mathrm{Im}(h \circ F_1) \leq \mathrm{Im}(F_2)$.

This is to say that a morphism of pairs is a morphism of the ambient objects A_1 and A_2 which takes $\mathrm{Im}(F_1)$ to $\mathrm{Im}(F_2)$.

Definition 83 (Category of pairs in $\mathscr{C}$)**.** Given a category $\mathscr{C}$ with all images the *category of pairs in* $\mathscr{C}$ (or the *category of* $\mathscr{C}$*-pairs*) is the category $\mathbf{Pair}(\mathscr{C})$ whose objects are $\mathscr{C}$-pairs, whose morphisms are morphisms of pairs in $\mathscr{C}$, for which the identity of (A, F) is id_A, and whose composition is given by composition of morphisms in $\mathscr{C}$.

We need that $\mathscr{C}$ has all images to show that $\mathbf{Pair}(\mathscr{C})$ is a category. If $\mathscr{C}$ doesn't have all images then morphisms of pairs may not be composable even if their underlying morphisms in $\mathscr{C}$ are composable.

Proposition 9. *Given a category $\mathscr{C}$ with all images we have that $\mathbf{Pair}(\mathscr{C})$ is a category.*

Proof. We show that morphisms compose. Suppose that $\mathbf{A}_i \coloneqq (A_i, F_i) \in \mathrm{Pair}(\mathscr{C})$ for each $i \in \{1, 2, 3\}$ and that $h_1 \colon \mathbf{A}_1 \to \mathbf{A}_2$ and $h_2 \colon \mathbf{A}_2 \to \mathbf{A}_3$ are morphisms of pairs. We show that $h_2 \circ h_1 \colon A_1 \to A_3$ is a morphism from $\mathbf{A}_1$ to $\mathbf{A}_3$. Since $\mathscr{C}$ has all images this is precisely the situation in Lemma 3 so morphisms in $\mathbf{Pair}(\mathscr{C})$ compose.

Again the associativity of composition and the identity property for $\mathbf{Pair}(\mathscr{C})$ follow immediately from those for $\mathscr{C}$. ☐

It is not surprising that the proof that $\mathbf{Pair}(\mathscr{C})$ is essentially identical to the proof that $\mathbf{Struct}^\rho$ is a category since by our characterization of the category $\mathbf{Fun}(\mathscr{I}_1, \mathscr{C})$ we find that $\mathbf{Struct}^\rho \cong \mathbf{Pair}(\mathscr{C})$.

From this isomorphism we see that given a structure $\mathbf{A} \coloneqq (A, F) \in \mathrm{Struct}^\rho$ we have that the relation $\mathbf{A}_N$ is the subobject $\mathrm{Im}(F)$ of A in $\mathscr{C}$ and that $\mathbf{A}$ has no nontrivial basic relators.

A.2.2 Hypergraphs

We examine the category of structures obtained from the n-hypergraph signature $\rho \colon \mathscr{I}_1 \to \mathbf{Fun}(\mathbf{Set}, \mathbf{Set})$.

Definition 84 (n-hypergraph)**.** Given a set A we refer to $\mathbf{A} \coloneqq (A, F)$ where $F \subset \binom{A}{\leq n}$ as an *n-hypergraph* on A.

We denote by Hyp_n the class of n-hypergraphs.

Definition 85 (Morphism of n-hypergraphs)**.** Given n-hypergraphs $\mathbf{A}_1 \coloneqq (A_1, F_1)$ and $\mathbf{A}_2 \coloneqq (A_2, F_2)$ we refer to a function $h \colon A_1 \to A_2$ as a *morphism* from $\mathbf{A}_1$ to $\mathbf{A}_2$ and write $h \colon \mathbf{A}_1 \to \mathbf{A}_2$ when $h(F_1) \subset F_2$ where

$$h(F_1) \coloneqq \{\, \{\, h(a) \mid a \in E \,\} \mid E \in F_1 \,\}.$$

Definition 86 (Category of n-hypergraphs)**.** We denote by $\mathbf{Hyp}_n$ the *category of n-hypergraphs* whose objects form the class Hyp_n, whose morphisms are morphisms of n-hypergraphs, for which the identity of (A, F) is id_A, and whose composition is given by composition of functions.

It is evident that $\mathbf{Struct}^\rho \cong \mathbf{Hyp}_n$.

A.3 The Yoneda embedding

Although our definition of structure appears to be more general than the structures usually considered in model theory, whose basic relations are subsets of Cartesian powers of the universe, we show that each category of **Set**-sourced structures embeds into a category of **Set**-structures whose basic relations are all subsets of Cartesian powers of the universe, at the expense that our new index category may be large where our original index category was small.

This will be useful to us because Cartesian powers are easier for us to analyze than general sets and because this embedding will help us bring the tools of homological algebra to bear on categories of **Set**-sourced structures in a natural way.

The driving device here is the Yoneda embedding. Given a locally small category $\mathscr{C}$ let $よ\_ \colon \mathscr{C} \to \mathbf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathbf{Set})$ denote the contravariant Hom-functor. Recall the following embedding of categories due to Yoneda.

Lemma 4 (Yoneda Lemma)**.** *Let $\mathscr{C}$ be a locally small category. The functor $よ\_$ is full and faithful.*

We actually need more general source categories than **Set**.

Definition 87 (Exponential category)**.** We say that a full subcategory $\mathscr{C}$ of **Set** is *exponential* when $\mathscr{C}$ is closed under taking subsets and forming exponential objects.

One example of an exponential category is the full subcategory of **Set** whose objects are the empty set and every singleton set. The largest possible example of an exponential category is **Set** itself. We will be most interested in the exponential category **FinSet** whose objects form the class FinSet of finite sets.

Given an exponential category $\mathscr{C}$ and a signature $\rho \colon \mathscr{I} \to \mathbf{Fun}(\mathscr{C}, \mathscr{D})$ we apply the Yoneda Lemma to $\mathbf{Struct}^\rho$ to obtain an embedding

$$よ\_ \colon \mathbf{Struct}^\rho \hookrightarrow \mathbf{Fun}((\mathbf{Struct}^\rho)^{\mathrm{op}}, \mathbf{Set}).$$

Each structure $\mathbf{A} \in \mathrm{Struct}^\rho_A$ determines a functor

$$よ_{\mathbf{A}} \colon (\mathbf{Struct}^\rho)^{\mathrm{op}} \to \mathbf{Set}\,.$$

If $\mathbf{A} \in \mathrm{Struct}^{\rho}_{A}$ and $\mathbf{C} \in \mathrm{Struct}^{\rho}_{C}$ with $A, C \in \mathrm{Ob}(\mathscr{C})$ then

$$よ_{\mathbf{A}}(\mathbf{C}) = \mathrm{Hom}(\mathbf{C}, \mathbf{A}) \subset A^{C}$$

so $よ_{\mathbf{A}}$ can be restricted on its codomain to a functor from $(\mathbf{Struct}^{\rho})^{\mathrm{op}}$ to $\mathscr{C}$.

Definition 88 (Exponential Yoneda functor)**.** Given an exponential category $\mathscr{C}$ and a signature $\rho\colon \mathscr{I} \to \mathbf{Fun}(\mathscr{C}, \mathscr{D})$ the *exponential Yoneda functor*

$$よ^{\mathscr{C}}_{\_}\colon \mathbf{Struct}^{\rho} \to \mathbf{Fun}((\mathbf{Struct}^{\rho})^{\mathrm{op}}, \mathscr{C})$$

of ρ over $\mathscr{C}$ is the functor obtained by restricting the codomain of $よ_{\mathbf{A}}$ to $\mathscr{C}$ for each $\mathbf{A} \in \mathrm{Struct}^{\rho}$.

Restricting the codomain of a functor preserves embeddings so the exponential Yoneda functor is also an embedding, which we are justified in calling the *exponential Yoneda embedding*.

Definition 89 (Yoneda signature)**.** Given an exponential category $\mathscr{C}$ and a signature $\rho\colon \mathscr{I} \to \mathbf{Fun}(\mathscr{C}, \mathscr{D})$ the *Yoneda signature* $\rho^{\mathscr{C}}$ of ρ over $\mathscr{C}$ is the functor $\rho^{\mathscr{C}}\colon (\mathbf{Struct}^{\rho})^{\mathrm{op}} \to \mathbf{Fun}(\mathscr{C}, \mathscr{C})$ defined as follows. For $\mathbf{C} \in \mathrm{Struct}^{\rho}_{C}$ the functor $\rho^{\mathscr{C}}(\mathbf{C})\colon \mathscr{C} \to \mathscr{C}$ is given by $\rho^{\mathscr{C}}(\mathbf{C})(A) \coloneqq A^{C}$ for each set A and

$$\rho^{\mathscr{C}}(\mathbf{C})(h) \coloneqq h \circ \_\colon A_1^{C} \to A_2^{C}$$

for each function $h\colon A_1 \to A_2$. Given a morphism $f\colon \mathbf{C}_2 \to \mathbf{C}_1$ in $\mathbf{Struct}^{\rho}$ the natural transformation $\rho^{\mathscr{C}}(f)\colon \rho^{\mathscr{C}}(\mathbf{C}_1) \to \rho^{\mathscr{C}}(\mathbf{C}_2)$ is given by

$$\rho^{\mathscr{C}}(f)_A \coloneqq \_ \circ f\colon \rho^{\mathscr{C}}(\mathbf{C}_1)(A) \to \rho^{\mathscr{C}}(\mathbf{C}_2)(A).$$

In order to show that $\rho^{\mathscr{C}}$ is a signature we need the following lemma about exponential categories.

Lemma 5. *Given an exponential category $\mathscr{C}$ and a function $h\colon A \to B$ in $\mathscr{C}$ we have that h has an image triple (V, θ, ψ) in $\mathscr{C}$ where θ is surjective.*

Proof. Since $\mathscr{C}$ is closed under taking subobjects we can form the subset $V \subset B$ given by $V \coloneqq \{\, b \in B \mid (\exists a \in A)(h(a) = b) \,\}$. Define $\theta\colon A \to V$ by $\theta(a) \coloneqq h(a)$ for each $a \in A$ and let $\psi\colon V \hookrightarrow B$ be the inclusion of V as a subset of B. Since $\mathscr{C}$ is a full subcategory of $\mathbf{Set}$ we have that (V, θ, ψ) remains an image triple for h in $\mathscr{C}$. Observe that θ is surjective. $\square$

Proposition 10. *The Yoneda signature $\rho^{\mathscr{C}}$ is a signature.*

Proof. We show that $\rho^{\mathscr{C}}$ is a functor. For $\mathbf{C} \in \mathrm{Struct}^{\rho}_{C}$ we show that $\rho^{\mathscr{C}}(\mathbf{C})\colon \mathscr{C} \to \mathscr{C}$ is a functor and hence an object of $\mathbf{Fun}(\mathscr{C},\mathscr{C})$.

Given $A \in \mathrm{Ob}(\mathscr{C})$ we have that $\rho^{\mathscr{C}}(\mathbf{C})(A) = A^C$, which is an object of $\mathscr{C}$ since $A, C \in \mathrm{Ob}(\mathscr{C})$ and $\mathscr{C}$ is an exponential category. Thus, $\rho^{\mathscr{C}}(\mathbf{C})$ takes objects to objects. Given a function $h\colon A_1 \to A_2$ in $\mathscr{C}$ we have that $\rho^{\mathscr{C}}(\mathbf{C})(h)\colon A_1^C \to A_2^C$ is a function. Thus, $\rho^{\mathscr{C}}(\mathbf{C})$ takes morphisms to morphisms. Given an identity map $\mathrm{id}_A\colon A \to A$ in $\mathscr{C}$ we have that

$$\rho^{\mathscr{C}}(\mathbf{C})(\mathrm{id}_A) = \mathrm{id}_A \circ \_ = \mathrm{id}_{A^C}\colon A^C \to A^C.$$

Thus, $\rho^{\mathscr{C}}(\mathbf{C})$ takes identities to identities. Given functions $h_1\colon A_1 \to A_2$ and $h_2\colon A_2 \to A_3$ in $\mathscr{C}$ we have that

$$\begin{aligned}
\rho^{\mathscr{C}}(\mathbf{C})(h_2 \circ h_1) &= (h_2 \circ h_1) \circ _ \\
&= h_2 \circ (h_1 \circ _) \\
&= (h_2 \circ _) \circ (h_1 \circ _) \\
&= \rho^{\mathscr{C}}(\mathbf{C})(h_2) \circ \rho^{\mathscr{C}}(\mathbf{C})(h_1)
\end{aligned}$$

so $\rho^{\mathscr{C}}(\mathbf{C})$ respects composition. We find that $\rho^{\mathscr{C}}(\mathbf{C})$ is a functor and hence $\rho^{\mathscr{C}}$ takes objects to objects.

For $f\colon \mathbf{C}_2 \to \mathbf{C}_1$ in $\mathbf{Struct}^{\rho}$ we show that $\rho^{\mathscr{C}}(f)\colon \rho^{\mathscr{C}}(\mathbf{C}_1) \to \rho^{\mathscr{C}}(\mathbf{C}_2)$ is a natural transformation and hence a morphism of $\mathbf{Fun}(\mathscr{C},\mathscr{C})$. Given a function $h\colon A_1 \to A_2$ in $\mathscr{C}$ it is immediate that

$$\rho^{\mathscr{C}}(f)_{A_2} \circ \rho^{\mathscr{C}}(\mathbf{C}_1)(h) = h \circ \_ \circ f = \rho^{\mathscr{C}}(\mathbf{C}_2)(h) \circ \rho^{\mathscr{C}}(f)_{A_1}$$

so $\rho^{\mathscr{C}}(f)$ is a natural transformation and hence $\rho^{\mathscr{C}}$ takes morphisms to morphisms.

Given an identity morphism $\mathrm{id}_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{C}$ and any $A \in \mathrm{Ob}(\mathscr{C})$ we have that $\rho^{\mathscr{C}}(\mathrm{id}_{\mathbf{C}})_A = \_ \circ \mathrm{id}_C = \mathrm{id}_{A^C}$ so $\rho^{\mathscr{C}}(\mathrm{id}_{\mathbf{C}}) = \mathrm{id}_{\rho^{\mathscr{C}}(\mathbf{C})}$ is the identity natural transformation of $\rho^{\mathscr{C}}(\mathbf{C})$ and $\rho^{\mathscr{C}}$ takes identities to identities.

Given morphisms $f_2\colon \mathbf{C}_3 \to \mathbf{C}_2$ and $f_1\colon \mathbf{C}_2 \to \mathbf{C}_1$ in $\mathbf{Struct}^{\rho}$ and $A \in \mathrm{Ob}(\mathscr{C})$ we have that

$$\rho^{\mathscr{C}}(f_2)_A \circ \rho^{\mathscr{C}}(f_1)_A = \_ \circ f_1 \circ f_2 = \rho^{\mathscr{C}}(f_1 \circ f_2)_A$$

so $\rho^{\mathscr{C}}$ respects composition and is thus a functor from $(\mathbf{Struct}^{\rho})^{\mathrm{op}}$ to $\mathbf{Fun}(\mathscr{C},\mathscr{C})$.

It remains to show that given a monomorphism $F\colon U \hookrightarrow \rho^{\mathscr{C}}_{A_1}$ in $\mathbf{Fun}((\mathbf{Struct}^{\rho})^{\mathrm{op}}, \mathscr{C})$ and any function $h\colon A_1 \to A_2$ in $\mathscr{C}$ we have that $\mathrm{Im}(\rho^{\mathscr{C}}_h \circ F)$ exists in $\mathbf{Fun}((\mathbf{Struct}^{\rho})^{\mathrm{op}}, \mathscr{C})$.

For each $\mathbf{C} \in \mathrm{Ob}(\mathscr{C})$ let $(V_{\mathbf{C}}, \theta_{\mathbf{C}}, \psi_{\mathbf{C}})$ be an image triple for $(\rho^{\mathscr{C}}_h \circ F)_{\mathbf{C}}$ where $\theta_{\mathbf{C}}$ is surjective as guaranteed by Lemma 5. For each morphism $f\colon \mathbf{C}_2 \to \mathbf{C}_1$ we have that

$$\begin{aligned}\mathrm{Im}(\rho^{\mathscr{C}}_{A_2}(f) \circ \psi_{\mathbf{C}_1}) &= \mathrm{Im}(\rho^{\mathscr{C}}_{A_2}(f) \circ \psi_{\mathbf{C}_1} \circ \theta_{\mathbf{C}_1}) \\ &= \mathrm{Im}(\psi_{\mathbf{C}_2} \circ \theta_{\mathbf{C}_2} \circ U(f)) \\ &\leq \mathrm{Im}(\psi_{\mathbf{C}_2})\end{aligned}$$

so the codomain restriction

$$g_f \coloneqq (\rho^{\mathscr{C}}_{A_2}(f) \circ \psi_{\mathbf{C}_1})|_{V_{\mathbf{C}_2}}\colon V_{\mathbf{C}_1} \to V_{\mathbf{C}_2}$$

of $\rho^{\mathscr{C}}_{A_2}(f) \circ \psi_{\mathbf{C}_1}$ to $V_{\mathbf{C}_2}$ exists in $\mathscr{C}$.

We use this data to factor $\rho^{\mathscr{C}}_h \circ F$. Define a functor $V\colon (\mathbf{Struct}^{\rho})^{\mathrm{op}} \to \mathscr{C}$ by $V(\mathbf{C}) \coloneqq V_{\mathbf{C}}$ for each $\mathbf{C} \in \mathrm{Struct}^{\rho}$ and $V(f) \coloneqq g_f$ for each morphism $f\colon \mathbf{C}_2 \to \mathbf{C}_1$ in $\mathbf{Struct}^{\rho}$. Define natural transformations $\theta\colon U \to V$ and $\psi\colon V \to \rho^{\mathscr{C}}_{A_2}$ whose components at $\mathbf{C}$ are $\theta_{\mathbf{C}}$ and $\psi_{\mathbf{C}}$, respectively.

We show that $V\colon (\mathbf{Struct}^{\rho})^{\mathrm{op}} \to \mathscr{C}$ is a functor. Given $\mathbf{C} \in \mathrm{Struct}^{\rho}$ we have that $V(\mathbf{C}) = V_{\mathbf{C}} \in \mathrm{Ob}(\mathscr{C})$ by definition of V so V takes objects to objects. Given a morphism $f\colon \mathbf{C}_2 \to \mathbf{C}_1$ in $\mathbf{Struct}^{\rho}$ we have that $V(f) = g_f$ is a morphism in $\mathscr{C}$ by definition so V takes morphisms to morphisms. Given an object $\mathbf{C} \in \mathrm{Struct}^{\rho}_C$ and its identity morphism $\mathrm{id}_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{C}$ we have that $\rho^{\mathscr{C}}$ is a presignature so by Proposition 6 we have that $\rho^{\mathscr{C}}_{A_2}$ is a functor and hence $\rho^{\mathscr{C}}_{A_2}(\mathrm{id}_{\mathbf{C}}) = \mathrm{id}_{\rho^{\mathscr{C}}_{A_2}(\mathbf{C})}$. It follows that so

$$V(\mathrm{id}_{\mathbf{C}}) = g_{\mathrm{id}_{\mathbf{C}}} = (\rho^{\mathscr{C}}_{A_2}(\mathrm{id}_{\mathbf{C}}) \circ \psi_{\mathbf{C}})|_{V_{\mathbf{C}}} = (\mathrm{id}_{\rho^{\mathscr{C}}_{A_2}(\mathbf{C})} \circ \psi_{\mathbf{C}})|_{V_{\mathbf{C}}} = \psi_{\mathbf{C}}|_{V_{\mathbf{C}}} = \mathrm{id}_{V(\mathbf{C})}$$

so V takes identities to identities. Given morphisms $f_2\colon \mathbf{C}_3 \to \mathbf{C}_2$ and $f_1\colon \mathbf{C}_2 \to \mathbf{C}_1$ in $\mathbf{Struct}^{\rho}$ we have that $\mathrm{Im}(\rho^{\mathscr{C}}_{A_2}(f_1) \circ \psi_{\mathbf{C}_1}) \leq \mathrm{Im}(\psi_{\mathbf{C}_2})$ and hence

$$\begin{aligned}V(f_1 \circ f_2) &= g_{f_1 \circ f_2} \\ &= (\rho^{\mathscr{C}}_{A_2}(f_1 \circ f_2) \circ \psi_{\mathbf{C}_1})|_{V_{\mathbf{C}_3}} \\ &= (\rho^{\mathscr{C}}_{A_2}(f_2) \circ \rho^{\mathscr{C}}_{A_2}(f_1) \circ \psi_{\mathbf{C}_1})|_{V_{\mathbf{C}_3}} \\ &= (\rho^{\mathscr{C}}_{A_2}(f_2) \circ \psi_{\mathbf{C}_2})|_{V_{\mathbf{C}_3}} \circ (\rho^{\mathscr{C}}_{A_2}(f_1) \circ \psi_{\mathbf{C}_1})|_{V_{\mathbf{C}_2}} \\ &= V(f_2) \circ V(f_1).\end{aligned}$$

Thus, V respects composition and is a functor from $(\mathbf{Struct}^{\rho})^{\mathrm{op}}$ to $\mathscr{C}$.

It is evident that θ and ψ are natural transformations.

Since each of the components of ψ are injective we have that ψ is monic. By definition we have that $\rho_h^{\mathscr{C}} \circ F = \psi \circ \theta$. It follows that $\mathrm{Im}(\rho^{\mathscr{C}} \circ F)$ exists in $\mathbf{Fun}((\mathbf{Struct}^\rho)^{\mathrm{op}}, \mathscr{C})$ and contains ψ. $\square$

We can use the exponential Yoneda embedding to obtain a functor from $\mathbf{Struct}^\rho$ to the category of $\mathscr{C}$-structures $\mathbf{Struct}^{\rho^{\mathscr{C}}}$.

Definition 90 (Cartesian Yoneda functor)**.** Given an exponential category $\mathscr{C}$ and a signature $\rho\colon \mathscr{I} \to \mathbf{Fun}(\mathscr{C}, \mathscr{D})$ the *Cartesian Yoneda functor*

$$\text{か}\colon \mathbf{Struct}^\rho \to \mathbf{Struct}^{\rho^{\mathscr{C}}}$$

is defined as follows. Given $\mathbf{A} \in \mathrm{Struct}_A^\rho$ we define $\text{か}(\mathbf{A})$ to be the subobject of $\rho_A^{\mathscr{C}}$ containing the monomorphism $F_{\mathbf{A}}\colon \text{よ}_{\mathbf{A}}^{\mathscr{C}} \hookrightarrow \rho_A^{\mathscr{C}}$ given by $(F_{\mathbf{A}})_{\mathbf{C}}\colon \text{よ}_{\mathbf{A}}^{\mathscr{C}}(\mathbf{C}) \to \rho_A^{\mathscr{C}}(\mathbf{C})$ where $(F_{\mathbf{A}})_{\mathbf{C}}(f) \coloneqq f$. Given $h\colon \mathbf{A}_1 \to \mathbf{A}_2$ we define $\text{か}(h) \coloneqq h$.

Intuitively, the Cartesian Yoneda functor is the natural inclusion of $\mathrm{Hom}(\mathbf{C}, \mathbf{A})$ into A^C.

We will need the following lemma about images in general categories which gives a sufficient condition for the image of structure under a morphism to be contained in another structure.

Lemma 6. *Let $\mathscr{C}$ be a category with $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$ and morphisms $\alpha\colon X \to Z$, $\beta\colon X \to Y$, and $\gamma\colon Y \hookrightarrow Z$ such that $\alpha = \gamma \circ \beta$. Let $U \in \mathrm{Ob}(\mathscr{C})$ with $\theta_X\colon X \to U$ and $\theta_Z\colon U \hookrightarrow Z$ a factorization of α witnessing that θ_Z is the image of α. We have that $\theta_Z \leq \gamma$.*

Proof. Since $\beta\colon X \to Y$ and $\gamma\colon Y \hookrightarrow Z$ form a factorization of α we have by definition of the image θ_Z that there exists a morphism $s\colon U \to Y$ such that $\theta_Z = \gamma \circ s$. Thus, $\theta_Z \leq \gamma$. $\square$

Proposition 11. *We have that $\text{か}\colon \mathbf{Struct}^\rho \to \mathbf{Struct}^{\rho^{\mathscr{C}}}$ is a functor.*

Proof. We show that か takes objects to objects. Given $\mathbf{A} \in \mathrm{Struct}_A^\rho$ we must show that $F_{\mathbf{A}}\colon \text{よ}_{\mathbf{A}}^{\mathscr{C}} \to \rho_A^{\mathscr{C}}$ is a monomorphism. Let $U\colon (\mathbf{Struct}^\rho)^{\mathrm{op}} \to \mathscr{C}$ be a functor and let $H_1, H_2\colon U \to \text{よ}_{\mathbf{A}}^{\mathscr{C}}$ be natural transformations. We show that $F_{\mathbf{A}} \circ H_1 = F_{\mathbf{A}} \circ H_2$ implies that $H_1 = H_2$.

Fix an object $\mathbf{C} \in \mathrm{Struct}_C^\rho$. We have that $(\text{よ}_{\mathbf{A}}^{\mathscr{C}})(\mathbf{C}) = \mathrm{Hom}(\mathbf{C}, \mathbf{A})$ and $(\rho_A^{\mathscr{C}})(\mathbf{C}) = A^C$. By definition we find that $(F_{\mathbf{A}})_{\mathbf{C}}\colon \mathrm{Hom}(\mathbf{C}, \mathbf{A}) \to A^C$ is the map taking a morphism $h\colon \mathbf{C} \to \mathbf{A}$ to its underlying function $h\colon C \to A$. We have that $U(\mathbf{C})$ is a set and $(H_1)_{\mathbf{C}}, (H_2)_{\mathbf{C}}\colon U(\mathbf{C}) \to \mathrm{Hom}(\mathbf{C}, \mathbf{A})$ are functions. Since $(F_{\mathbf{A}})_{\mathbf{C}}$ is injective we have that $(H_1)_{\mathbf{C}} = (H_2)_{\mathbf{C}}$. As the components of H_1 and

H_2 are the same we have that $H_1 = H_2$. Thus, $F_{\mathbf{A}}$ is a monomorphism and か does take objects of $\mathbf{Struct}^{\rho}$ to objects of $\mathbf{Struct}^{\rho^{\mathscr{C}}}$.

We show that か takes morphisms to morphisms. Given $\mathbf{A}_1 \in \mathrm{Struct}^{\rho}_{A_1}$, $\mathbf{A}_2 \in \mathrm{Struct}^{\rho}_{A_2}$, and a morphism $h\colon \mathbf{A}_1 \to \mathbf{A}_2$ we must show that

$$\text{か}(h)(\text{か}(\mathbf{A}_1)) \leq \text{か}(\mathbf{A}_2)$$

as subobjects of $\rho^{\mathscr{C}}_{A_2}$.

Choose representative monomorphisms $F_{\mathbf{A}_1}$ and $F_{\mathbf{A}_2}$ of $\text{か}(\mathbf{A}_1)$ and $\text{か}(\mathbf{A}_2)$, respectively. By our lemma it suffices to show that there exists a natural transformation $\eta\colon \text{よ}^{\mathscr{C}}_{\mathbf{A}_1} \to \text{よ}^{\mathscr{C}}_{\mathbf{A}_2}$ such that $\rho^{\mathscr{C}}_h \circ F_{\mathbf{A}_1} = F_{\mathbf{A}_2} \circ \eta$. We claim that we can take $\eta = \text{よ}^{\mathscr{C}}_h$. Unraveling definitions we find that for each $\mathbf{C} \in \mathrm{Struct}^{\rho}$ and each $f \in \text{よ}^{\mathscr{C}}_{\mathbf{A}_1}(\mathbf{C})$ we have

$$(\rho^{\mathscr{C}}_h \circ F_{\mathbf{A}_1})_{\mathbf{C}}(f) = h \circ f = (F_{\mathbf{A}_2} \circ \text{よ}^{\mathscr{C}}_h)_{\mathbf{C}}(f),$$

as claimed.

We show that か takes identities to identities. Given $\mathbf{A} \in \mathrm{Struct}^{\rho}_{A}$ and $\mathrm{id}_{\mathbf{A}}\colon \mathbf{A} \to \mathbf{A}$ we have that $\text{か}(\mathrm{id}_{\mathbf{A}}) = \mathrm{id}_A$, which is the identity morphism of $\text{か}(\mathbf{A})$ in $\mathbf{Struct}^{\rho^{\mathscr{C}}}$.

We show that か respects the composition of morphisms. Since か maps a morphism $h\colon \mathbf{A}_1 \to \mathbf{A}_2$ to its underlying function $h\colon A_1 \to A_2$ and in both $\mathbf{Struct}^{\rho}$ and $\mathbf{Struct}^{\rho^{\mathscr{C}}}$ composition of morphisms is given by composition of the underlying functions we have that か respects composition. $\square$

The Cartesian Yoneda functor is an embedding of categories.

Proposition 12. *The functor* $\text{か}\colon \mathbf{Struct}^{\rho} \to \mathbf{Struct}^{\rho^{\mathscr{C}}}$ *is full and faithful.*

Proof. We show that か is full. Suppose that $h \in \mathrm{Hom}(\text{か}(\mathbf{A}_1), \text{か}(\mathbf{A}_2))$. By definition we have that $\rho^{\mathscr{C}}(h)(\text{か}(\mathbf{A}_1)) \leq \text{か}(\mathbf{A}_2)$ so there exists some $\eta\colon \text{よ}^{\mathscr{C}}_{\mathbf{A}_1} \to \text{よ}^{\mathscr{C}}_{\mathbf{A}_2}$ such that $\rho^{\mathscr{C}}_h \circ F_{\mathbf{A}_1} = F_{\mathbf{A}_2} \circ \eta$ for representative monomorphisms $F_{\mathbf{A}_1}$ and $F_{\mathbf{A}_2}$ of $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively. Since $\mathrm{id}_{A_1} \in \mathrm{Hom}(\mathbf{A}_1, \mathbf{A}_1)$ this implies that

$$h = h \circ \mathrm{id}_{A_1} = (\rho^{\mathscr{C}}_h \circ F_{\mathbf{A}_1})_{\mathbf{A}_1}(\mathrm{id}_{A_1}) = (F_{\mathbf{A}_2} \circ \eta)_{\mathbf{A}_1}(\mathrm{id}_{A_1})$$

from which it follows that h is in the image of

$$(F_{\mathbf{A}_2})_{\mathbf{A}_1}\colon \mathrm{Hom}(\mathbf{A}_1, \mathbf{A}_2) \to A_2^{A_1}.$$

Thus, $h \in \mathrm{Hom}(\mathbf{A}_1, \mathbf{A}_2)$.

Note that since $\text{か}(h) \coloneqq h$ we have that か is faithful. $\square$

We are thus justified in referring to か as the *Cartesian Yoneda embedding.* As a special case, we have proven our remark from the beginning of this section. Given a signature $\rho\colon \mathscr{I} \to \mathbf{Fun}(\mathbf{Set}, \mathscr{D})$ we have that か is an embedding of $\mathbf{Struct}^{\rho}$ into $\mathbf{Struct}^{\rho^{\mathbf{Set}}}$. Given objects $\mathbf{A} \in \mathrm{Struct}^{\rho}_{A}$ and $\mathbf{C} \in \mathrm{Struct}^{\rho}_{A}$ the relation of $\text{か}(\mathbf{A})$ at $\mathbf{C}$ is a subset of A^{C}, or a C-ary relation on A in the classical sense.