Category theory background for constraint satisfaction (Part 2)

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Introduction

- These are notes on the category theory background needed to read: Maximilian Hadek, Tomáš Jakl, and Jakub Opršal. "A categorical perspective on constraint satisfaction: The wonderland of adjunctions." In: *arXiv e-prints* (Mar. 2025). arXiv: 2503.10353 [cs.LO]

Introduction

- Relational structures as presheaves
- Nerve of a category
- Nerve of a functor
- Discrete Grothendieck construction
- Kan extensions

Relational structures as presheaves

- A combinatorial graph may be thought of as a set V of vertices, a set E of edges, and a pair of maps $s\colon E \to V$ and $t\colon E \to V$ where $s(e)$ and $t(e)$ are the source and target of the directed edge e.
- This can be thought of as a functor from the diagram

$$V \underset{t}{\overset{s}{\leftleftarrows}} E$$

to the category Set.

Relational structures as presheaves

- We could also think of a graph as a contravariant functor from

$$V \underset{t}{\overset{s}{\rightrightarrows}} E$$

  to the category Set.
- That is, a functor $A\colon \mathcal{S}^{\mathrm{op}} \to \mathrm{Set}$.

Relational structures as presheaves

- We introduce the opposite category in order to think of a graph as a presheaf.
- A $\mathscr{D}$-valued presheaf on $\mathscr{C}$ is a functor of the form $F\colon \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$.
- These generalize sheaves from geometry.
- A standard example is the sheaf of smooth functions on a manifold M, where $\mathscr{C}$ is the lattice of open sets of M, $\mathscr{D}$ is the category of rings, and $F(U)$ is the ring of smooth functions on an open set U.

Relational structures as presheaves

- We can view a (multiply-sorted) relational structure as a functor $A\colon \mathcal{S}^{\mathrm{op}} \to \mathrm{Set}$ in a similar manner by taking $\mathcal{S}$ to have one object for each relation and one object for each universe.
- Morphisms specify components, as indicated in the binary case for graphs.

Nerve of a category

- Categories can be thought of as structures in this way.
- Let Δ be the category whose objects are the finite chains $[n] = \{0, \ldots, n-1\}$ for each $n \in \mathbb{N}$ and whose morphisms are isotone maps.
- A *simplicial set* is a functor $A \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$.

Nerve of a category

- Given a category $\mathscr{C}$, we may define a functor (the *nerve* of $\mathscr{C}$) $N(\mathscr{C})\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ by setting $(N(\mathscr{C}))([n])$ to be the set of all sequences of $n-1$ composable morphisms

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} A_{n-1}$$

  in $\mathscr{C}$.
- The morphisms of Δ allow us to see the composition and identities of $\mathscr{C}$ in $N(\mathscr{C})$.

Nerve of a functor

- Recall that for any category $\mathscr{C}$ we have the covariant Yoneda embedding
$$よ_{\_} : \mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$$
given by
$$よ_A(B) = \mathscr{C}(B, A)$$
for objects $A, B \in \mathrm{Ob}(\mathscr{C})$.

Nerve of a functor

- Given a functor $F\colon \mathscr{C} \to \mathscr{D}$, the *nerve* of F is the functor

$$N_F\colon \mathscr{D} \to [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$$

given by

$$N_F(A)\colon \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$$

where

$$N_F(A) = よ_A \circ F^{\mathrm{op}}.$$

Nerve of a functor

- For an object B of $\mathscr{C}$ this means that

$$(N_F(A))(B) = \mathscr{D}(F(B), A).$$

- There is a typo in this definition in the paper.
- What does this have to do with the nerve $N_{\mathscr{C}}$ of a category $\mathscr{C}$?

Nerve of a functor

- Let $F\colon \Delta \to \mathrm{Cat}$ be the inclusion functor from the simplex category Δ to the category of categories Cat.
- In this case we have that the nerve of F is a functor

$$N_F\colon \mathrm{Cat} \to [\Delta^{\mathrm{op}}, \mathrm{Set}].$$

Nerve of a functor

- By definition we have that

$$N_F(\mathscr{C})\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$$

is given by

$$(N_F(\mathscr{C}))([n]) = \mathrm{Cat}(F([n]), \mathscr{C}) = [[n], \mathscr{C}].$$

- This says that

$$(N_F(\mathscr{C}))([n]) = (N(\mathscr{C}))([n]).$$

- We find that $N_F(\mathscr{C}) = N(\mathscr{C})$, so this generalizes the nerve construction.

Nerve of a functor

- What would it mean for N_F to have a left adjoint in this case?
- Suppose that $G \dashv N_F$ with $G\colon [\Delta^{\mathrm{op}}, \mathrm{Set}] \to \mathrm{Cat}$.
- We have a natural bijection

$$\mathrm{Cat}(G(X), Y) \cong [\Delta^{\mathrm{op}}, \mathrm{Set}](X, N_F(Y))$$

where X is a simplicial set and Y is a category.

Nerve of a functor

- This means that we have a natural bijection

$$[G(X), Y] \cong [\Delta^{\mathrm{op}}, \mathrm{Set}](X, N_F(Y)).$$

- That is, functors from the category $G(X)$ made from the simplicial set X are in bijective correspondence with simplicial set morphisms from X to the nerve of the category Y.

Nerve of a functor

- This means that we have a natural bijection

$$[G(X), Y] \cong [\Delta^{\mathrm{op}}, \mathrm{Set}](X, N_F(Y)).$$

- The category $G(X)$ is the category freely determined by the "diagram" X. It is the most general category containing morphisms whose composition obeys the rules indicated by X.
- One might say $G(X)$ is the "free category" over X.

Discrete Grothendieck construction

Definition (Grothendieck construction)

Let $F\colon \mathscr{C} \to \mathrm{Set}$ be a functor. Define $\int F$ to be the category where $\mathrm{Ob}(\int(F))$ consists of pairs (s, a) where $s \in \mathrm{Ob}(\mathscr{C})$ and $a \in F(s)$ and

$$\left(\int F\right)((s, a), (t, b)) = \{\, f\colon s \to t \mid (F(f))(a) = b \,\}\,.$$

The *Grothendieck construction* $\mathrm{gr}(F)$ is the functor

$$\mathrm{gr}(F)\colon \int F \to \mathscr{C}$$

given by $(\mathrm{gr}(F))(s, a) = s$.

Discrete Grothendieck construction

- Let's consider the case of a graph $F\colon \mathcal{S}^{\text{op}} \to \text{Set}$.
- The category $\int F$ has objects (V, v) where $v \in F(V)$ and (E, e) where $e \in F(E)$.
- There are morphisms $(E, e) \to (V, e)$ which send edges to their sources and targets.
- The functor $\text{gr}(F)$ tells us whether an object is a vertex or an edge.

Kan extensions

Definition ((Global) Kan extension)

Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a functor. Given another category $\mathscr{D}$, let

$$F^*\colon [\mathscr{C}', \mathscr{D}] \to [\mathscr{C}, \mathscr{D}]$$

be given by

$$F^*(G) = G \circ F.$$

A left adjoint to F^* is the *left Kan extension* Lan_F along F. A right adjoint to F^* is the *right Kan extension* Ran_F along F.

- That is, $F^* = よ_{\mathscr{D}}(F)$ and

$$\mathrm{Lan}_F \dashv よ_{\mathscr{D}}(F) \dashv \mathrm{Ran}_F\,.$$

Kan extensions

- The existence of Lan_F means that there is a bijection

$$[\mathscr{C}', \mathscr{D}](\mathrm{Lan}_F(X), Y) \cong [\mathscr{C}, \mathscr{D}](X, F^*(Y))$$

  which is natural in functors $X \colon \mathscr{C} \to \mathscr{D}$ and $Y \colon \mathscr{C}' \to \mathscr{D}$.
- We could fix a functor $X \colon \mathscr{C} \to \mathscr{D}$ and ask that there exists a functor $\mathrm{Lan}_F(X) \colon \mathscr{C} \to \mathscr{D}$ such that this isomorphism is still natural in Y, even if the left adjoint of F^* doesn't exist.

Kan extensions

- A functor $F\colon \mathscr{C} \to \mathscr{D}$ has a colimit if and only if $\mathrm{Lan}_K(F)$ along $K\colon \mathscr{C} \to 1$ exists. The colimit is $\mathrm{colim}(F) = (\mathrm{Lan}_K(F))(*)$ where $*$ is the sole object of 1.
- Similarly, $\lim(F) = (\mathrm{Ran}_K(F))(*)$ when it exists.
- The existence of an adjoint can also be expressed in terms of the existence of a particular Kan extension.

Kan extensions

Lemma

Given a finite category $\mathscr{C}$, a functor $F\colon \mathscr{C} \to \mathscr{D}$, and a finitely complete category $\mathscr{E}$ we have that $よ_{\mathscr{E}}(F) \dashv \mathrm{Ran}_F$ exists. If $\mathscr{E}$ is finitely cocomplete then we have the analogous conclusion for Lan_F. Both adjoints exist for $\mathscr{E} = \mathrm{Fin}$.