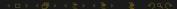
Category theory background for constraint satisfaction (Part 2)

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Introduction

■ These are notes on the category theory background needed to read: Maximilian Hadek, Tomáš Jakl, and Jakub Opršal. "A categorical perspective on constraint satisfaction: The wonderland of adjunctions." In: arXiv e-prints (Mar. 2025). arXiv: 2503.10353 [cs.L0]

Introduction

- Relational structures as presheaves
- Nerve of a category
- Nerve of a functor
- Discrete Grothendieck construction
- Kan extensions

- A combinatorial graph may be thought of as a set V of vertices, a set E of edges, and a pair of maps $s\colon E\to V$ and $t\colon E\to V$ where s(e) and t(e) are the source and target of the directed edge e.
- This can be thought of as a functor from the diagram

$$V \stackrel{s}{\longleftarrow} E$$

to the category Set.

■ We could also think of a graph as a contravariant functor from

$$V \xrightarrow{s} E$$

to the category Set.

■ That is, a functor $A : \mathcal{S}^{\mathrm{op}} \to \mathrm{Set}$.

- We introduce the opposite category in order to think of a graph as a presheaf.
- A \mathscr{D} -valued presheaf on \mathscr{C} is a functor of the form $F \colon \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$.
- These generalize sheaves from geometry.
- A standard example is the sheaf of smooth functions on a manifold M, where $\mathscr C$ is the lattice of open sets of M, $\mathscr D$ is the category of rings, and F(U) is the ring of smooth functions on an open set U.

- We can view a (multiply-sorted) relational structure as a functor $A \colon \mathcal{S}^{\mathrm{op}} \to \mathrm{Set}$ in a similar manner by taking \mathcal{S} to have one object for each relation and one object for each universe.
- Morphisms specify components, as indicated in the binary case for graphs.

Nerve of a category

- Categories can be thought of as structures in this way.
- Let Δ be the category whose objects are the finite chains $[n]=\{0,\ldots,n-1\}$ for each $n\in\mathbb{N}$ and whose morphisms are isotone maps.
- A simplicial set is a functor $A : \Delta^{op} \to Set$.

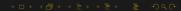
Nerve of a category

■ Given a category $\mathscr C$, we may define a functor (the *nerve* of $\mathscr C$) $N(\mathscr C)\colon \Delta^{\mathrm{op}}\to \mathrm{Set}$ by setting $(N(\mathscr C))([n])$ to be the set of all sequences of n-1 composable morphisms

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} A_{n-1}$$

in \mathscr{C} .

■ The morphisms of Δ allow us to see the composition and identities of $\mathscr C$ in $N(\mathscr C)$.



 \blacksquare Recall that for any category $\mathscr C$ we have the covariant Yoneda embedding

$${\sharp}_\colon \mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$$

given by

$$\sharp_A(B) = \mathscr{C}(B, A)$$

for objects $A, B \in Ob(\mathscr{C})$.

■ Given a functor $F: \mathscr{C} \to \mathscr{D}$, the *nerve* of F is the functor

$$N_F \colon \mathscr{D} \to [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$$

given by

$$N_F(A) \colon \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$$

where

$$N_F(A) = \sharp_A \circ F^{\mathrm{op}}.$$

■ For an object B of $\mathscr C$ this means that

$$(N_F(A))(B) = \mathscr{D}(F(B), A).$$

- There is a typo in this definition in the paper.
- What does this have to do with the nerve $N_{\mathscr{C}}$ of a category \mathscr{C} ?

- Let $F: \Delta \to \operatorname{Cat}$ be the inclusion functor from the simplex category Δ to the category of categories Cat .
- In this case we have that the nerve of F is a functor

$$N_F \colon \mathrm{Cat} \to [\Delta^{\mathrm{op}}, \mathrm{Set}].$$

By definition we have that

$$N_F(\mathscr{C}) \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$$

is given by

$$(N_F(\mathscr{C}))([n]) = \operatorname{Cat}(F([n]),\mathscr{C}) = [[n],\mathscr{C}].$$

This says that

$$(N_F(\mathscr{C}))([n]) = (N(\mathscr{C}))([n]).$$

■ We find that $N_F(\mathscr{C}) = N(\mathscr{C})$, so this generalizes the nerve construction.



- What would it mean for N_F to have a left adjoint in this case?
- Suppose that $G \dashv N_F$ with $G : [\Delta^{op}, Set] \to Cat$.
- We have a natural bijection

$$Cat(G(X), Y) \cong [\Delta^{op}, Set](X, N_F(Y))$$

where X is a simplicial set and Y is a category.

■ This means that we have a natural bijection

$$[G(X), Y] \cong [\Delta^{\mathrm{op}}, \mathrm{Set}](X, N_F(Y)).$$

■ That is, functors from the category G(X) made from the simplicial set X are in bijective correspondence with simplicial set morphisms from X to the nerve of the category Y.

■ This means that we have a natural bijection

$$[G(X), Y] \cong [\Delta^{\mathrm{op}}, \mathrm{Set}](X, N_F(Y)).$$

- The category G(X) is the category freely determined by the "diagram" X. It is the most general category containing morphisms whose composition obeys the rules indicated by X.
- lacksquare One might say G(X) is the "free category" over X.

Discrete Grothendieck construction

Definition (Grothendieck construction)

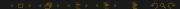
Let $F\colon \mathscr{C} \to \mathrm{Set}$ be a functor. Define $\int F$ to be the category where $\mathrm{Ob}(\int(F))$ consists of pairs (s,a) where $s\in \mathrm{Ob}(\mathscr{C})$ and $a\in F(s)$ and

$$\left(\int F \right) ((s, a), (t, b)) = \{ f \colon s \to t \mid (F(f))(a) = b \}.$$

The *Grothendieck construction* gr(F) is the functor

$$\operatorname{gr}(F) \colon \int F \to \mathscr{C}$$

given by (gr(F))(s, a) = s.



Discrete Grothendieck construction

- Let's consider the case of a graph $F: \mathcal{S}^{\mathrm{op}} \to \mathrm{Set}$.
- The category $\int F$ has objects (V, v) where $v \in F(V)$ and (E, e) where $e \in F(E)$.
- There are morphisms $(E,e) \rightarrow (V,e)$ which send edges to their sources and targets.
- The functor gr(F) tells us whether an object is a vertex or an edge.

Definition ((Global) Kan extension)

Let $F \colon \mathscr{C} \to \mathscr{C}'$ be a functor. Given another category \mathscr{D} , let

$$F^* \colon [\mathscr{C}', \mathscr{D}] \to [\mathscr{C}, \mathscr{D}]$$

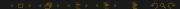
be given by

$$F^*(G) = G \circ F.$$

A left adjoint to F^* is the left Kan extension Lan_F along F. A right adjoint to F^* is the right Kan extension Ran_F along F.

lacksquare That is, $F^* = \mbox{$\sharp$}_{\mathscr{D}}(F)$ and

$$\operatorname{Lan}_F \dashv \sharp_{\mathscr{D}}(F) \dashv \operatorname{Ran}_F$$
.



lacksquare The existence of ${
m Lan}_F$ means that there is a bijection

$$[\mathscr{C}',\mathscr{D}](\operatorname{Lan}_F(X),Y)\cong [\mathscr{C},\mathscr{D}](X,F^*(Y))$$

which is natural in functors $X \colon \mathscr{C} \to \mathscr{D}$ and $Y \colon \mathscr{C}' \to \mathscr{D}$.

■ We could fix a functor $X \colon \mathscr{C} \to \mathscr{D}$ and ask that there exists a functor $\operatorname{Lan}_F(X) \colon \mathscr{C} \to \mathscr{D}$ such that this isomorphism is still natural in Y, even if the left adjoint of F^* doesn't exist.

- A functor $F \colon \mathscr{C} \to \mathscr{D}$ has a colimit if and only if $\operatorname{Lan}_K(F)$ along $K \colon \mathscr{C} \to 1$ exists. The colimit is $\operatorname{colim}(F) = (\operatorname{Lan}_K(F))(*)$ where * is the sole object of 1.
- Similarly, $\lim(F) = (\operatorname{Ran}_K(F))(*)$ when it exists.
- The existence of an adjoint can also be expressed in terms of the existence of a particular Kan extension.

Lemma

Given a finite category $\mathscr C$, a functor $F\colon \mathscr C\to \mathscr D$, and a finitely complete category $\mathscr E$ we have that $\sharp_{\mathscr E}(F)\dashv \mathrm{Ran}_F$ exists. If $\mathscr E$ is finitely cocomplete then we have the analogous conclusion for Lan_F . Both adjoints exist for $\mathscr E=\mathrm{Fin}$.