Category theory background for constraint satisfaction (Part 1)

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Introduction

- These are notes on the category theory background needed to read: Maximilian Hadek, Tomáš Jakl, and Jakub Opršal. “A categorical perspective on constraint satisfaction: The wonderland of adjunctions”. In: *arXiv e-prints* (Mar. 2025). arXiv: 2503.10353 [cs.LO]

Introduction

- PCSP in categorical terms
- Reductions as adjunctions
- Polymorphisms

PCSP in categorical terms

Definition (Promise (decision) problem)

A *promise (decision) problem* over some class C of elements, called *instances* is a pair of subclasses $Y, N \subset C$. We refer to members of Y as *YES instances* and members of N as *NO instances*.

- We say that a promise problem is *well-defined* when Y and N are disjoint. We usually assume this.
- We may write (C, Y, N) to denote a promise problem.

PCSP in categorical terms

Definition (Reduction)

A *reduction* from a promise problem (C, Y, N) to (C', Y', N') is a function $f\colon C \to C'$ such that $f(Y) \subset Y'$ and $f(N) \subset N'$.

- We will usually focus on *tractable* promise problems, which are those that have polynomial-time algorithm for deciding whether a given instance is in the YES or NO class.
- We are therefore concerned with *efficient* reductions, which are those that can be computed in polynomial time, as these preserve tractability.

PCSP in categorical terms

- Given objects A and B of a category $\mathscr{C}$, we denote by $A \to B$ the existence of a morphism from A to B in $\mathscr{C}$.
- Similarly, we denote by $A \not\to B$ the absence of such a morphism.
- Note that in categorical logic we would write $A \vdash B$ rather than $A \to B$ and $A \nvdash B$ when $A \not\to B$.
- This is distinct from the material implication $A \implies B$, which is usually realized as an internal hom bifunctor.

PCSP in categorical terms

Definition (Promise CSP)

Let A and B be objects in the category $\mathscr{C}$. A *promise CSP* for the template (A, B) is is the promise decision problem whose instances are $\mathrm{Ob}(\mathscr{C})$, whose `YES` instances are objects I with $I \to A$, and whose `NO` instances are object I with $I \not\to B$.

- We might write $\mathrm{PCSP}_{\mathscr{C}}(A, B) = (\mathrm{Ob}(\mathscr{C}), Y(A), N(B))$ to indicate this promise decision problem.
- We might write $\mathrm{PCSP}(A, B)$ as a shorthand when the category $\mathscr{C}$ is clear from context.

PCSP in categorical terms

Definition (Thin category)

A *thin* (or *posetal*) category is a category $\mathscr{C}$ in which $|\mathscr{C}(A, B)| \leq 1$ for every pair of objects $A, B \in \mathscr{C}$.

- Thin categories are basically preorders (partial orders without antisymmetry).

PCSP in categorical terms

- Given a category $\mathscr{C}$, let $\mathrm{Thin}(\mathscr{C})$ be the the category whose objects are those of $\mathscr{C}$ and whose morphisms are given by setting $(\mathrm{Thin}(\mathscr{C}))(A, B)$ to be a singleton set when $A \to B$ and setting $(\mathrm{Thin}(\mathscr{C}))(A, B) = \varnothing$ when $A \not\to B$.
- Note that $\mathrm{Thin}(\mathrm{Set})$ is equivalent to 2, the walking arrow category.
- The category $\mathrm{Thin}(\mathscr{C})$ is also known as the *preorder reflection* of $\mathscr{C}$.

PCSP in categorical terms

- Note that for each

  $$\mathrm{PCSP}_{\mathscr{C}}(A, B) = (\mathrm{Ob}(\mathscr{C}), Y(A), N(B))$$

  we can define

  $$\mathrm{PCSP}_{\mathrm{Thin}(\mathscr{C})}(A, B) = (\mathrm{Ob}(\mathrm{Thin}(\mathscr{C})), Y(A), N(B)).$$

- Claim: The identity map $1_{\mathrm{Ob}(\mathscr{C})}$ is a reduction from $\mathrm{PCSP}_{\mathscr{C}}(A, B)$ to $\mathrm{PCSP}_{\mathrm{Thin}(\mathscr{C})}(A, B)$ as well as a reduction in the other direction.

PCSP in categorical terms

- We might say that $\mathrm{PCSP}_{\mathscr{C}}(A, B)$ is isomorphic to $\mathrm{PCSP}_{\mathrm{Thin}(\mathscr{C})}(A, B)$.
- Since $\mathrm{Thin}(\mathscr{C})$ is a preorder, it looks like promise constraint isn't "really" a categorical notion.
- In practice we can't make use of this reduction since it's equivalent to being able to solve $\mathrm{PCSP}_{\mathscr{C}}(A, B)$.

Reductions as adjunctions

Definition (Adjunction)

Given functors $L\colon \mathscr{C} \to \mathscr{D}$ and $R\colon \mathscr{D} \to \mathscr{C}$ we say that (L, R) is an *adjoint pair* (with *left adjoint* L and *right adjoint* R) when

$$\mathscr{D}(L(X), Y) \cong \mathscr{C}(X, R(Y))$$

is a natural isomorphism of bifunctors $\mathscr{C}^{\mathrm{op}} \times \mathscr{D} \to \mathrm{Set}$.

- A critical example is the adjunction $F \dashv U$ between the free and forgetful functors for a variety of algebras.

Reductions as adjunctions

- Another important example is $\Sigma \dashv \Delta \dashv \Pi$ where $\Delta \colon \mathscr{C} \to \mathscr{C}^2$ is the diagonal functor, Σ is the coproduct, and Π is the product.
- More general limits and colimits may be realized as adjoints in a similar fashion.

Reductions as adjunctions

Definition (Adjunction)

Given functors $L\colon \mathscr{C} \to \mathscr{D}$ and $R\colon \mathscr{D} \to \mathscr{C}$ we say that (L, R) is an *adjoint pair* when there exist natural transformations $\epsilon\colon L \circ R \to 1_{\mathscr{D}}$ and $\eta\colon 1_{\mathscr{C}} \to R \circ L$ such that

$$1_L = \epsilon 1_L \circ 1_L \eta$$

and

$$1_R = 1_R \epsilon \circ \eta 1_R.$$

We call ϵ and η the *counit* and *unit* of the adjunction, respectively.

Reductions as adjunctions

- We can obtain reductions from adjunctions.

Lemma

Whenever $L\colon \mathscr{C} \to \mathscr{D}$ is a functor and $L \dashv R$ we have that L is a reduction from $\mathrm{PCSP}_{\mathscr{C}}(A, B)$ to $\mathrm{PCSP}_{\mathscr{D}}(A', B')$ if and only if $A \to R(A')$ and $R(B') \to B$.

Reductions as adjunctions

Proof.

Suppose L is a reduction. Since $A \to A$ we have that A is a YES instance for $\mathrm{PCSP}_{\mathscr{C}}(A, B)$. This means that $L(A)$ is a YES instance for $\mathrm{PCSP}_{\mathscr{D}}(A', B')$. It follows that $L(A) \to A'$, which implies that $A \to R(A')$.

Since $\epsilon\colon L \circ R \to 1_{\mathscr{D}}$ we have that $(L \circ R)(B') \to B'$. This means that $L(R(B'))$ is not a NO instance for $\mathrm{PCSP}_{\mathscr{D}}(A', B')$. It follows that $R(B')$ is not a NO instance for $\mathrm{PCSP}_{\mathscr{C}}(A, B)$. That is, $R(B') \to B$.

Reductions as adjunctions

Proof (Cont.)

Now suppose that $A \to R(A')$ and $R(B') \to B$ but we don't know that L is a reduction.
Given a YES instance I of $\mathrm{PCSP}_{\mathscr{C}}(A, B)$ we have $I \to A$. Since $A \to R(A')$ this implies that $I \to R(A')$. Since $L \vdash R$ we have that $L(I) \to A'$, so $L(I)$ is a YES instance of $\mathrm{PCSP}_{\mathscr{D}}(A', B')$.
Given a NO instance I of $\mathrm{PCSP}_{\mathscr{C}}(A, B)$ we have $I \not\to B$. If L failed to preserve NO instances then we would have $L(I) \to B'$ for some such I, which implies that $I \to R(B')$ and hence $I \to B$, a contradiction. □

Reductions as adjunctions

- There is a remark in the paper that the CSP literature uses the notion of "thin adjunction", which means that $L(X) \to Y$ if and only if $X \to R(Y)$.
- This appears to just be the usual notion of adjunction between $\mathrm{Thin}(\mathscr{C})$ and $\mathrm{Thin}(\mathscr{D})$, as opposed to an adjunction between $\mathscr{C}$ and $\mathscr{D}$.

Polymorphisms

Definition (Polymorphism of an object)

Given an object A in a category $\mathscr{C}$ with finite products and some $n \in \mathbb{N}$, an n-ary *polymorphism* of A is a $\mathscr{C}$-morphism $A^n \to A$.

Definition (Polymorphism of a template)

Given objects A and B in a category $\mathscr{C}$ with finite products and some $n \in \mathbb{N}$, an n-ary polymorphism of (A, B) is a $\mathscr{C}$-morphism $A^n \to B$.

Polymorphisms

- Each template (A, B) in a category $\mathscr{C}$ has a corresponding *polymorphism minion* $\mathrm{Pol}(A, B)\colon \mathrm{Fin} \to \mathrm{Fin}$ given by $n \mapsto \mathscr{C}(A^n, B)$.
- Efficient reductions between PCSPs come from *minion homomorphisms*, which are natural transformations between minions.

Polymorphisms

- At the bottom of page 3 it is claimed that polymorphisms $A^n \to B$ do not form an algebra, but it seems like they do under generalized composition, as long as one includes the projections $A^m \to A$ and $B^m \to B$.
- It is similarly claimed that $\mathrm{Pol}(A, B)$ is not a monad (although I'm not 100% sure on what category), but that $\mathrm{Pol}(A)$ is. It looks to me like both $\mathrm{Pol}(A)$ and $\mathrm{Pol}(A, B)$ are multiply-sorted algebraic structures which correspond to monads.