

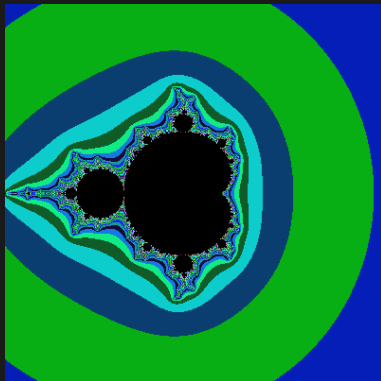
Holomorphy over real algebras

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PALS at CU Boulder
2026 October 13

Introduction

- The Mandelbrot set is the set of points $c \in \mathbb{C}$ for which the orbit of 0 under iteration of the mapping $z \mapsto z^2 + c$ does not escape to infinity.



Introduction

- A *real algebra* is a real vector space endowed with a bilinear multiplication operation.
- For any real algebra $\mathbb{A}$ we can consider the analogue of the Mandelbrot set. This is the set of points $c \in \mathbb{A}$ for which the orbit of 0 under iteration of the mapping $x \mapsto x^2 + c$ does not escape to infinity.
- I was particularly interested in the Mandelbrot set analogue for the *RPSions*, the 3-dimensional real algebra $\mathbb{J} = \mathbb{R}[r, p, s]$ whose multiplication is the bilinear extension of the rock-paper-scissors operation.

Introduction

- I'll show an animation of the Mandelbrot set analogue for $\mathbb{J}$.
- Link to the animation: `https://aten.cool/documents/slides/RPS\_animation.gif`

Defining holomorphy

- The study of this generalized Mandelbrot set belongs to hypercomplex dynamics.
- In order to generalize the basic definitions of complex dynamics to arbitrary real algebras, we should like to have some generalized notion of a complex-differentiable, or holomorphic, function.
- Let $\mathbb{A}$ be a real algebra and let $f: \mathbb{A} \rightarrow \mathbb{A}$ be a function. We would like to define

$$f'(x) = \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{f(x + hv) - f(x)}{hv},$$

but in general it doesn't make sense to divide by $v \in \mathbb{A}$.

Defining holomorphy

- Note that we could ask for

$$vf'(x) = \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{f(x + hv) - f(x)}{h} = \nabla_v f(x)$$

or

$$f'(x)v = \nabla_v f(x),$$

since multiplication need not be commutative.

- Here we'll ask for both (i.e., that $f'(x)$ is central in $\mathbb{A}$) and that there is a unique choice of $f'(x)$ which works for all v .

Defining holomorphy

Definition (Holomorphic function)

Given a finite-dimensional real algebra $\mathbb{A}$ and a function $f: \mathbb{A} \rightarrow \mathbb{A}$, we say that f is *holomorphic* at a point $x \in \mathbb{A}$ when

- 1 f is differentiable at x and
- 2 there exists a unique $\delta \in \mathbb{A}$ such that for all directions $v \in \mathbb{A}$ we have that $v\delta = \delta v = \nabla_v f(x)$.

When f is holomorphic at x we denote this unique value δ by $f'(x)$.

First examples of holomorphy

- Holomorphic functions over $\mathbb{R}$ (by our definition), are precisely the differentiable functions.
- Our definition agrees with the usual one for $\mathbb{C}$.
- Since the center of $\mathbb{H}$ (the quaternions) is $\mathbb{R}$, any holomorphic function over $\mathbb{H}$ must have a purely real derivative.
- Hamilton himself observed that it seemed difficult to endow the quaternions with an analogue of differentiation which makes $x \mapsto x^2$ holomorphic.

First examples of holomorphy

- Holomorphic functions over $\mathbb{R}[\mathbb{Z}/2\mathbb{Z}]$ are those functions $f(xe + ya) = u(x, y)e + v(x, y)a$ such that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

First examples of holomorphy

- If this were a more analytic talk I would explain how I know that holomorphic functions over $\mathbb{J} = \mathbb{R}[r, p, s]$ are those functions

$$f(xr + yp + zs) = u(x, y, z)r + v(x, y, z)p + w(x, y, z)s$$

such that

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial z} = \frac{\partial w}{\partial x} = 0$$

$$2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}$$

$$2\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$2\frac{\partial w}{\partial y} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

First properties of holomorphy

- Recall that $x \in \mathbb{A}$ is said to *annihilate* $\mathbb{A}$ when for all $y \in \mathbb{A}$ we have that $xy = yx = 0$.

Proposition

Let $\mathbb{A}$ be a finite-dimensional real algebra. The following are equivalent:

- 1 All constant functions $f: \mathbb{A} \rightarrow \mathbb{A}$ are holomorphic.
- 2 No nonzero element of $\mathbb{A}$ annihilates $\mathbb{A}$.

First properties of holomorphy

- Any constant function has $\nabla_v f(x) = 0$ so we need that

$$v\delta = \delta v = 0$$

for all v , which means that δ must annihilate $\mathbb{A}$.

- Note that $\delta = 0$ is already a solution, so there is a unique choice for δ if and only if there is no nonzero element of $\mathbb{A}$ which annihilates $\mathbb{A}$.
- We have $f'(x) = 0$ for constant functions.

First properties of holomorphy

- Recall that $e \in \mathbb{A}$ is an *identity element* for $\mathbb{A}$ when $ex = xe = x$ for all $x \in \mathbb{A}$.

Proposition

Let $\mathbb{A}$ be a finite-dimensional real algebra. The following are equivalent:

- 1 *The identity function $f: \mathbb{A} \rightarrow \mathbb{A}$ given by $f(x) = x$ is holomorphic.*
- 2 *There exists an identity element $e \in \mathbb{A}$.*

First properties of holomorphy

- For each $v \in \mathbb{A}$ we must have that

$$v\delta = \delta v = \nabla_v f(x) = v,$$

so $f'(x) = e$.

First properties of holomorphy

- Given an element $\alpha \in \mathbb{A}$, let $\lambda_\alpha: \mathbb{A} \rightarrow \mathbb{A}$ be given by $\lambda_\alpha(x) = \alpha x$ and let $\rho_\alpha: \mathbb{A} \rightarrow \mathbb{A}$ be given by $\rho_\alpha(x) = x\alpha$.
- We say that α is *divisible* in $\mathbb{A}$ when at least one of λ_α and ρ_α is invertible.

First properties of holomorphy

Proposition

Suppose that $\mathbb{A}$ is a finite-dimensional real algebra which has a divisible element α . Given holomorphic functions $f, g: \mathbb{A} \rightarrow \mathbb{A}$ we have that the sum $f + g: \mathbb{A} \rightarrow \mathbb{A}$ given by
$$(f + g)(x) = f(x) + g(x)$$
is holomorphic with
$$(f + g)'(x) = f'(x) + g'(x).$$

First properties of holomorphy

- We must have

$$v\delta = \delta v = \nabla_v(f + g)(x) = \nabla_v f(x) + \nabla_v g(x),$$

so $\delta = f'(x) + g'(x)$ is a possible solution.

- When we choose the direction $v = \alpha$ we find that this solution is forced to be the only one.

First properties of holomorphy

Proposition

Suppose that $\mathbb{A}$ is a finite-dimensional real algebra which has a divisible element α , that $f : \mathbb{A} \rightarrow \mathbb{A}$ is holomorphic, and that $c \in \mathbb{R}$. We have that the function $cf : \mathbb{A} \rightarrow \mathbb{A}$ given by $(cf)(x) = cf(x)$ is holomorphic with $(cf)'(x) = cf'(x)$.

- Thus, the holomorphic functions over such a real algebra form a real vector space in the expected way.

First properties of holomorphy

Proposition

Suppose that $\mathbb{A}$ contains a divisible element and let $a \in \mathbb{A}$. We have that $\lambda_a: \mathbb{A} \rightarrow \mathbb{A}$ is holomorphic if and only if a is central.

- We must have that

$$v\delta = \delta v = \nabla_v \lambda_a(x) = av,$$

so the divisible element forces the only solution to be $f'(x) = a$.

The Product Rule

- Given $a \in \mathbb{A}$ we say that a is *middle nuclear* when for all $x, y \in \mathbb{A}$ we have that $x(ay) = (xa)y$ and we say that $a \in \mathbb{A}$ is *communal* when for all $x, y \in \mathbb{A}$ we have that $x(ay) = (ay)x$.

Proposition

Suppose that $\mathbb{A}$ contains a divisible element. The following are equivalent:

- 1 *The Product Rule holds in $\mathbb{A}$. That is, for all holomorphic functions $f, g: \mathbb{A} \rightarrow \mathbb{A}$ we have that the function $fg: \mathbb{A} \rightarrow \mathbb{A}$ given by $(fg)(x) = f(x)g(x)$ is again holomorphic with $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.*
- 2 *Every central element of $\mathbb{A}$ is middle nuclear and communal.*

The Product Rule

- Let $f(x) = ax$ with a central and let $g(x) = b$.
- We must have

$$v\delta = \delta v = \nabla_v(fg)(x) = (\nabla_v f(x))g(x) + f(x)\nabla_v g(x).$$

- We know that $\nabla_v f(x) = vf'(x) = va$ and that $\nabla_v g(x) = vg'(x) = 0$ so we find that

$$v(ab) = (ab)v = (va)b + (ax)0 = (va)b.$$

The Product Rule

- For the other direction, we need to solve

$$v\delta = \delta v = \nabla_v(fg)(x) = (\nabla_v f(x))g(x) + f(x)\nabla_v g(x).$$

- This becomes

$$v\delta = \delta v = (vf'(x))g(x) + f(x)(vg'(x))$$

- Since $f'(x)$ and $g'(x)$ are central we have that

$$\begin{aligned}(vf'(x))g(x) + f(x)(vg'(x)) &= v(f'(x)g(x)) + f(x)(g'(x)v) \\ &= v(f'(x)g(x)) + (f(x)g'(x))v \\ &= v(f'(x)g(x) + f(x)g'(x)) \\ &= (f'(x)g(x) + f(x)g'(x))v.\end{aligned}$$

The Product Rule

Corollary

Suppose that $\mathbb{A}$ contains an identity element. The following are equivalent:

- 1 The Product Rule holds in $\mathbb{A}$.*
- 2 The real algebra $\mathbb{A}$ is commutative and associative.*

The Chain Rule

- Given $a \in \mathbb{A}$ we say that a is *weakly middle nuclear* when for all $x \in \mathbb{A}$ and all $y \in C_{\mathbb{A}}$ we have that $x(ay) = (xa)y$ and we say that $a \in \mathbb{A}$ is *weakly communal* when for all $x \in \mathbb{A}$ and all $y \in C_{\mathbb{A}}$ we have that $x(ay) = (ay)x$.

Proposition

Suppose that $\mathbb{A}$ contains a divisible element. The following are equivalent:

- 1 *The Chain Rule holds in $\mathbb{A}$. That is, for all holomorphic functions $f, g: \mathbb{A} \rightarrow \mathbb{A}$ we have that the function $g \circ f: \mathbb{A} \rightarrow \mathbb{A}$ given by $(g \circ f)(x) = g(f(x))$ is again holomorphic with $(g \circ f)'(x) = f'(x)g'(f(x))$.*
- 2 *Every central element of $\mathbb{A}$ is weakly middle nuclear and weakly communal.*

The Chain Rule

- Since the Chain Rule is equivalent to an algebraic condition which is a weaker version of the algebraic condition for the Product Rule, we see that, under the assumption of a divisible element, any real algebra where the Product Rule holds must satisfy the Chain Rule, as well.

Conclusion

- While the condition

$$v\delta = \delta v = \nabla_v f(x)$$

certainly connects multiplication in $\mathbb{A}$ with directional derivatives of f , it still feels surprising that so many familiar properties of differentiation could be expressed as first-order properties of $\mathbb{A}$.

Conclusion

- One could imagine considering a weaker condition like

$$v\delta = \nabla_v f(x)$$

or generalizing to something like

$$p_1(v, \delta) = \cdots = p_k(v, \delta) = \nabla_v f(x)$$

for a family of polynomials $\{p_1, \dots, p_k\}$ over $\mathbb{A}$.

- It seems like these analogues of holomorphy shouldn't have such a direct connection to algebraic properties of $\mathbb{A}$, but I can't at the moment articulate why the choice made here behaves so well.