Equational logic for hyperalgebras and $\mathbb{F}_1$

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2025 August 24

Introduction

- Multiply-valued operations
- Wealds
- Modules over wealds
- Equational logic

Multiply-valued operations

- Given sets A and B, a *bosk* $f\colon A \lessdot B$ from A to B is a function $f_\star\colon A \to \mathcal{P}(B)$.
- Each bosk $f\colon A \lessdot B$ has a corresponding relation $f^\star \subset A \times B$ given by
$$f^\star = \{\, (a, b) \in A \times B \mid b \in f_\star(a) \,\}\,.$$

Multiply-valued operations

- Given a set A and some $n \in \mathbb{N}$, we say that a bosk $f \colon A^n \Yleft A$ is a *propagation* of the set A.
- Propagations include operations, hyperoperations, and partial operations as special cases.

Multiply-valued operations

- Given bosks $f\colon A \lessdot B$ and $g\colon B \lessdot C$, we have a *composite* bosk $g \circ f\colon A \lessdot C$ given by

$$(g \circ f)_\star(a) = \bigcup_{b \in f_\star(a)} g_\star(b).$$

- This is equivalent to defining $g \circ f\colon A \lessdot C$ by

$$(g \circ f)^\star = f^\star \circ g^\star.$$

Multiply-valued operations

- Given a propagation $f\colon A^m \lessdot A$ and m propagations $g_1, \ldots, g_m\colon A^n \lessdot A$, we define the *generalized composite*

$$f[g_1, \ldots, g_m]\colon A^n \lessdot A$$

by

$$(f[g_1, \ldots, g_m])_\star(a_1, \ldots, a_n) = \bigcup \left\{ f_\star(b_1, \ldots, b_m) \,\middle|\, (b_1, \ldots, b_m) \in \prod_{i=1}^{m} (g_i)_\star(a_1, \ldots, a_n) \right\}.$$

Multiply-valued operations

- An *almuqaba* is like an algebra but with propagations instead of operations.
- A *bale* is an almuqaba $(A, *, {}^{-1}, e)$ of signature $(2, 1, 0)$ such that
  1. (associativity) $x(yz) = (xy)z$ for all $x, y, z \in A$,
  2. (identity) $xe = ex \subset x$ for all $x \in A$, and
  3. (inverses) $xx^{-1} = x^{-1}x \subset e$ for all $x \in A$.
- Bales include semigroups, monoids, groups, and categories as special cases.

Wealds

- A *weald* is an almuqaba $(R, +, *, -, ^{-1}, 0, 1)$ such that
  1. $(R, +, -, 0)$ is a commutative bale,
  2. $(R, *, ^{-1}, 1)$ is a bale,
  3. $x(y + z) = xy + xz$ for all $x, y, z \in R$, and
  4. $(x + y)z = xz + yz$ for all $x, y, z \in R$.
- If a weald is an algebra, then it is the trivial ring.
- However, rings, fields, division rings, rigs, etc. may be viewed as wealds by allowing some basic propagations which are not operations on R.

Wealds

- The weald $\mathbb{F}_1$ is the singleton set $\{1\}$ where $1+1=\varnothing$, $1*1=1$, $-1=\varnothing$, $1^{-1}=1$, $0()=\varnothing$, and $1()=1$.
- Given a group $(G,*,{}^{-1},1)$, we can define a weald $(G,+,*,-,{}^{-1},0,1)$ by setting $x+y=-x=0()=\varnothing$ for all $x,y\in G$.
- We might identify the "field extension" $\mathbb{F}_{1^n}$ with the cyclic group $\mathbb{Z}/n\mathbb{Z}$ viewed as a weald.

Modules over wealds

Definition (Module over a weald)

Given a weald $\mathbf{R} = (R, +, *, -, ^{-1}, 0, 1)$, we say that an almuqaba $(M, +, -, 0, R)$ is a *module* over $\mathbf{R}$ (or an $\mathbf{R}$-*module*) when

1. $(M, +, -, 0)$ is a commutative bale,
2. for each $r \in R$ we have a unary propagation $r\colon M \lessdot M$,
3. $r(x + y) = rx + ry$ for all $r \in R$ and all $x, y \in M$,
4. $(r + s)x = rx + sx$ for all $r, s \in R$ and all $x \in M$,
5. $r(sx) = (rs)x$ for all $r, s \in R$ and all $x \in M$, and
6. $1x \subset x$ for all $x \in M$.

Modules over wealds

- In the case where **R** is a ring and $(M, +, -, 0)$ is an Abelian group, we recover the usual definition of a module over a ring.

Modules over wealds

- Let M be a set. Define $x + y = -x = 0() = \varnothing$ for $x, y \in M$. That is, let $(M, +, -, 0)$ be a commutative bale whose operations are all empty.
- We can endow $(M, +, -, 0)$ with the structure of an $\mathbb{F}_1$-module by setting $1x = x$ for all $x \in M$.

Modules over wealds

- Viewing sets as $\mathbb{F}_1$-modules in this way, we can view both the usual binomial coefficients and their q-analogues for finite fields $\mathbb{F}_q$ as special cases of the same construction.
- We also have other basic combinatorial properties, like the Vandermonde identity.

Modules over wealds

Definition (Disjoint union of modules)

Given a weald $\mathbf{R}$ and $\mathbf{R}$-modules $\mathbf{M} = (M, +, -, 0)$ and $\mathbf{N} = (N, +, -, 0)$, the *disjoint union* of $\mathbf{M}$ and $\mathbf{N}$ is the $\mathbf{R}$-module

$$\mathbf{M} \boxplus \mathbf{N} = (M \uplus N, +, -, 0)$$

where

$$x_M + y_M = (x + y)_M,$$

$$x_N + y_N = (x + y)_N,$$

$$x_M + y_N = \varnothing,$$

$$-(x_M) = (-x)_M \text{ and } -(x_N) = (-x)_N,$$

$$0() = 0_M() \uplus 0_N(),$$

and

$$rx_M = (rx)_M \text{ and } rx_N = (rx)_N.$$

Modules over wealds

- Given a module $\mathbf{M}$, let

$$\mathrm{GL}_n(\mathbf{M}) = \mathrm{Aut}(\mathbf{M}^n)$$

  and let

$$\mathrm{CL}_n(\mathbf{M}) = \mathrm{Aut}(n\mathbf{M}) = \mathrm{Aut}(\mathbf{M} \boxplus \cdots \boxplus \mathbf{M}).$$

- We have that $\mathrm{GL}_n(\mathbb{F}_1)$ is trivial, but $\mathrm{CL}_n(\mathbb{F}_1) = S_n$.
- We also have $\mathrm{CL}_n(\mathbb{F}) = F^\times \wr S_n$, so $\mathrm{CL}_n(\mathbb{F}_3) = C_n$ is the hyperoctahedral group.

Modules over wealds

$$
\begin{array}{ccccc}
S_n \cong \mathrm{CL}_n(\mathbb{F}_1) & \xrightarrow{\alpha} & \mathrm{CL}_n(\mathbb{F}) & \xrightarrow{\mathrm{CL}_n(\sigma)} & \mathrm{CL}_n(\mathbb{K}) \\
\downarrow \beta & & \downarrow \eta_{\mathbb{F}} & & \downarrow \eta_{\mathbb{K}} \\
S_1 \cong \mathrm{GL}_n(\mathbb{F}_1) & \xrightarrow{\gamma} & \mathrm{GL}_n(\mathbb{F}) & \xrightarrow{\mathrm{GL}_n(\sigma)} & \mathrm{GL}_n(\mathbb{K})
\end{array}
$$

Equational logic

- Our almuqabas certainly aren't algebras in the usual sense of universal algebra.
- In universal algebra, the identity $xx \approx x$ implies $(xy)(xy) \approx xy$, but this does not hold for all binary propagations.
- While terms and identities look the same for algebras and almuqabas, the inference rules are different.

Equational logic

- In a category with finite products, we can view a morphism $A^n \to A$ as an n-ary operation on A.
- The idea of categorical universal algebra is to view an algebraic theory as a category $\mathcal{T}$ whose objects are natural numbers.
- A morphism $m \to n$ represents an n-tuple of m-ary operations.
- A (classical) model of $\mathcal{T}$ is a (finite-product-preserving) functor from $\mathcal{T}$ to Set.

Equational logic

- Almuqabas aren't models of an algebraic theory $\mathcal{T}$ in Set.
- They also aren't models of an algebraic theory $\mathcal{T}$ in Rel.
- This is because the product in Rel is the disjoint union.
- In Rel, the Cartesian product is just some monoidal product.

Equational logic

Models	algebras	algebras for an operad
Multicategory	clone	operad
Category	Lawvere theory	PRO
Bifunctor	product	monoidal product
Identities	all	strongly regular
Inference rules	application, substitution	application

Equational logic

- Almuqabas aren't models of an algebraic theory $\mathcal{T}$ in Set.
- They also aren't models of an algebraic theory $\mathcal{T}$ in Rel.
- This is because the product in Rel is the disjoint union.
- In Rel, the Cartesian product is ~~just some monoidal product~~ a monoidal product with projections and pairing.
- That is, the Cartesian product is a product except without the uniqueness part of the universal property.

Equational logic

Models	algebras	almuqabas	algebras for an operad
Multicategory	clone	lush operad	operad
Category	Lawvere theory	lush PRO	PRO
Bifunctor	product	lush monoidal product	monoidal product
Identities	all	all	strongly regular
Inference rules	application, substitution	application, variable identification	application

Thank you!