A high school algebra problem

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Natural numbers

- I'll refer to the numbers 1, 2, 3, and so forth as the *natural numbers*.
- I won't consider 0 a natural number.
- By high school we are quite familiar with how to add, multiply, and exponentiate these numbers.
- You might even feel that you already know everything you can about their arithmetic.

The restricted high school identities

There are six basic identities which hold for all natural numbers x, y, and z. We call this collection of identities the *restricted high school identities* $\widehat{\mathrm{HSI}}$.

1. $x + y = y + x$
2. $x + (y + z) = (x + y) + z$
3. $x \cdot 1 = x$
4. $x \cdot y = y \cdot x$
5. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
6. $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

Polynomials

- Using $\widehat{\mathrm{HSI}}$ we can convert any expression involving addition and multiplication into a polynomial.
- For example, $(x \cdot (y + (x + 1))) + x$ yields $xy + x^2 + 2x$.
- We can check whether an equation like

$$(x + y)^2 - xy = x(x + y) + y^2$$

is true for all choices of x and y by completely expanding both sides using $\widehat{\mathrm{HSI}}$.

Polynomials

- It is never the case that two polynomials are equal for all choices of the variables x and y unless they are literally the same polynomial.
- For instance, something like

$$1000x^2 + xy = y^5 + 2024x$$

can't be true for all values of x and y unless both sides are the same fully-expanded polynomial.

Polynomials

- This means that all “rules of arithmetic” involving addition and multiplication can be deduced from $\widehat{\mathrm{HSI}}$.
- Thus, no *exotic identities* involving addition and multiplication can hold for the natural numbers.

Dedekind's high school identities

In 1888, Richard Dedekind gave a list of basic identities which hold for all natural numbers x, y, and z. They consist of $\widehat{\mathrm{HSI}}$ as well as the following five identities involving exponentiation. We call this collection of identities the *high school identities* HSI.

1. $1^x = 1$
2. $x^1 = x$
3. $x^{y+z} = x^y \cdot x^z$
4. $(x \cdot y)^z = x^z \cdot y^z$
5. $(x^y)^z = x^{y \cdot z}$

Tarski's High School Algebra Problem

- Since $\widehat{\mathrm{HSI}}$ can be used to derive every true identity involving addition and multiplication, it is natural to ask whether every true identity involving addition, multiplication, and exponentiation follows from HSI.
- This is the question that Alfred Tarski asked in the 1960s, which we call *Tarski's High School Algebra Problem*.
- This is equivalent to knowing whether there exist *exotic identities*, identities are true but which cannot be proven using HSI.

Exponential polynomials?

- Can we use HSI to expand any formula using addition, multiplication, and exponentiation into some kind of "exponential polynomial"?
- Observe that
$$(x^y)^z = x^{yz} = (x^z)^y$$
and
$$(x^2 + 3x + 2)^y = (x+1)^y(x+2)^y.$$
- It is not so clear what should count as being completely expanded.

Wilkie's exotic identity

- It turns out exotic identities involving exponentiation exist.
- In the early 1980s, Alex Wilkie produced the first example of an exotic identity, solving Tarski's High School Algebra Problem.
- This *Wilkie identity* $W(x, y)$ is

$$((1+x)^y + (1+x+x^2)^y)^x \cdot ((1+x^3)^x + (1+x^2+x^4)^x)^y =$$
$$((1+x)^x + (1+x+x^2)^x)^y \cdot ((1+x^3)^y + (1+x^2+x^4)^y)^x.$$

Wilkie's exotic identity

- We can prove that equations like the Wilkie identity are true using techniques from calculus on the corresponding functions on the real numbers.
- The relevant technique was pioneered by G.H. Hardy in the 1910s.
- For the Wilkie identity itself, we can "cheat" by using subtraction to factor $(1-x+x^2)^{xy}$ from both sides in order to see that it's true.

Wilkie's exotic identity

- Once we know that the Wilkie identity is true, it's still possible that it can be obtained from HSI by some complicated argument.
- In order to know it is really a new basic rule of arithmetic, we have to show that it can't be proven using HSI.
- How do we prove that you can't prove something?

Wilkie's exotic identity

- Wilkie's proof that HSI does not imply $W(x, y)$ was somewhat abstract, but in 1985 Gurevič gave a more concrete proof.

Wilkie's exotic identity

- Imagine we invented a new number system, with some collection of "numbers" and rules for addition, multiplication, and exponentiation such that the rules from HSI hold.
- For example, if our only "numbers" are 1 and 2, then the following operations work just like high school arithmetic.

Wilkie's exotic identity

$+$	1	2
1	2	1
2	1	2

$\cdot$	1	2
1	1	2
2	2	2

$\uparrow$	1	2
1	1	1
2	2	2

Wilkie's exotic identity

- If we could prove $W(x, y)$ from HSI, then every number system for which the HSI are true would have $W(x, y)$ true also.
- Thus, if we can make an example where HSI hold, but $W(x, y)$ does not, then we will have proven that $W(x, y)$ is not a consequence of the HSI.

Wilkie's exotic identity

- Unfortunately, $W(x, y)$ holds for the example whose only "numbers" are 1 and 2 that I gave before, so that one won't help us answer this question.

Wilkie's exotic identity

- Gurevič found a number system with 59 "numbers" for which the HSI hold, but $W(x, y)$ is not true.
- Other examples have been found, the smallest known (as of 2001) has 12 elements.

A small counterexample to Tarski's problem

$+$	1	2	3	4	a	b	c	d	e	f	g	h
1	2	3	4	4	2	3	d	3	3	3	3	4
2	3	4	4	4	3	4	3	4	4	4	4	4
3	4	4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4	4	4
a	2	3	4	4	b	4	b	3	h	3	3	4
b	3	4	4	4	4	4	4	4	4	4	4	4
c	d	3	4	4	b	4	b	3	3	3	3	4
d	3	4	4	4	3	4	3	4	4	4	4	4
e	3	4	4	4	h	4	3	4	4	3	h	4
f	3	4	4	4	3	4	3	4	3	4	3	4
g	3	4	4	4	3	4	3	4	h	3	4	4
h	4	4	4	4	4	4	4	4	4	4	4	4

A small counterexample to Tarski's problem

$\times$	1	2	3	4	a	b	c	d	e	f	g	h
1	1	2	3	4	a	b	c	d	e	f	g	h
2	2	4	4	4	b	4	b	4	4	4	4	4
3	3	4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4	4	4
a	a	b	4	4	c	b	c	b	h	4	4	4
b	b	4	4	4	b	4	b	4	4	4	4	4
c	c	b	4	4	c	b	c	b	4	4	4	4
d	d	4	4	4	b	4	b	4	4	4	4	4
e	e	4	4	4	h	4	4	4	4	4	h	4
f	f	4	4	4	4	4	4	4	4	4	4	4
g	g	4	4	4	4	4	4	4	h	4	4	4
h	h	4	4	4	4	4	4	4	4	4	4	4

A small counterexample to Tarski's problem

$\uparrow$	1	2	3	4	a	b	c	d	e	f	g	h
1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	4	4	4	4	4	4	4	f	4	4	4
3	3	4	4	4	e	4	4	4	g	4	e	h
4	4	4	4	4	4	4	4	4	4	4	4	4
a	a	c	c	c	c	c	c	c	c	c	c	c
b	b	4	4	4	4	4	4	4	4	4	4	4
c	c	c	c	c	c	c	c	c	c	c	c	c
d	d	4	4	4	f	4	4	4	4	4	4	4
e	e	4	4	4	4	4	4	4	h	4	4	4
f	f	4	4	4	4	4	4	4	4	4	4	4
g	g	4	4	4	h	4	4	4	4	4	h	4
h	h	4	4	4	4	4	4	4	4	4	4	4

A longer list of basic identities

- One might wonder whether adding $W(x, y)$ to HSI would make a new list of identities from which every true identity for the natural numbers (involving addition, multiplication, and exponentiation) can be proven.
- This is not the case.
- In 1990 Gurevič showed that there cannot be any finite list of such identities.

References

- Stanley N. Burris and Karen A. Yeats. *The Saga of the High School Identities*. 2004
- Clifford Bergman. *Universal Algebra: Fundamentals and Selected Topics*. Chapman and Hall/CRC, 2011. ISBN: 978-1-4398-5129-6

Thank you.