THE TOPOLOGY OF MAGMAS

CHARLOTTE ATEN

Introduction

A *magma* is an algebraic structure (S, f) consisting of an underlying set S and a single binary operation $f\colon S^2 \to S$. Much is known about specific families of magmas (semigroups, monoids, groups, semilattices, quasigroups, etc.) as well as magmas in general as treated in universal algebra. We seek to relate the study of magmas to the study of corresponding geometric objects. In order to do this we first analyze unary operations by way of their graphs. We show how function composition can be encoded by matrix multiplication, then generalize this to binary function composition. We characterize the spectra of the graphs of unary operations, show that all such graphs are planar, and present some initial results on the corresponding constructions for magmas.

Contents

Date: 2017.

email: caten2@u.rochester.edu

This research was conducted during the summer of 2015 with the support the United States federal McNair Program for Post-baccalaureate Achievement under the advisement of University of Rochester mathematics professors Jonathan Pakianathan and Mark Herman and during the summer of 2016 with the support of a National Science Foundation grant under the advisement of Iowa State University mathematics professors Clifford Bergman and William DeMeo.

1. Unary Operations

Before we tackle magmas we examine the case of unary operations. We restrict ourselves to the single-sorted situation, so our unary operations are all of the form $f\colon S \to S$ for some underlying set S.

1.1. **Operation Digraphs.**

We can view a unary operation as a set

$$\{(s, f(s)) \mid s \in S\}.$$

This set can be seen as the edge set of a digraph.

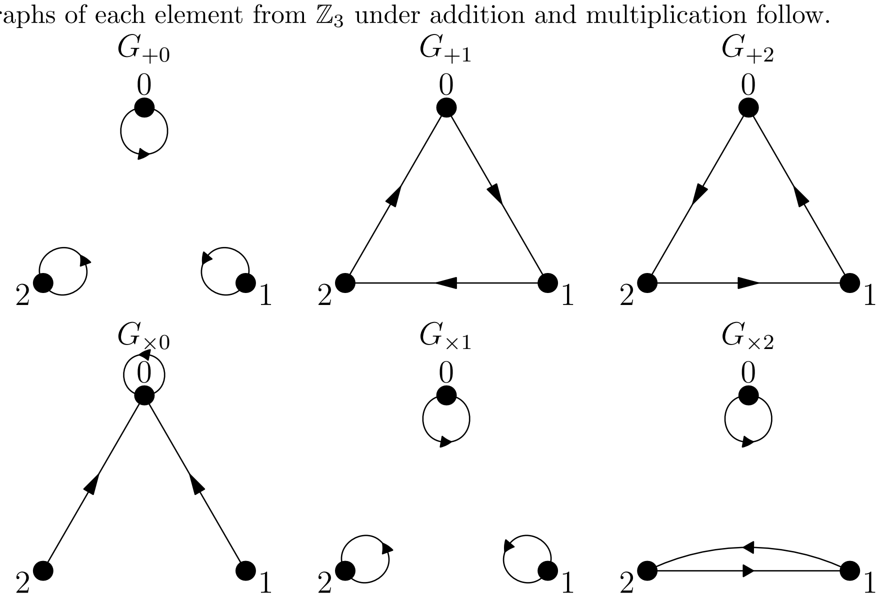
Definition (Operation digraph)**.** Let $f\colon S \to S$ be a unary operation. The *operation digraph* (or *functional digraph*[16, section 1.4]) of f, written G_f, is given by $G_f = G(S, E)$ where

$$E = \{(s, f(s)) \mid s \in S\}.$$

We can obtain unary operations from binary operations by fixing one of the arguments.

Definition. Let $f\colon S^2 \to S$ be a binary operation and let $s \in S$. The *left operation digraph* of s under f, written G^L_{fs}, is the operation digraph of $f^L_s\colon S \to S$ where $f^L_s(x) \coloneqq f(s, x)$ for $x \in S$. The *right operation digraph* of s under f, written G^R_{fs}, is defined analogously.

Naturally these graphs are identical in the case that f is commutative, allowing us to safely drop the superscript and simply speak of the *operation digraph* in question. This is the case for both addition and multiplication over $\mathbb{Z}_3$. The operation digraphs of each element from $\mathbb{Z}_3$ under addition and multiplication follow.



Such graphs appear in many contexts in mathematics. One can find them in the theory of semigroups[6], which deals in part with sets of functions from a set to itself. They are also studied at the intersection of number theory and dynamics[3]. The reader familiar with group theory will note the obvious connection between operation digraphs and Cayley graphs[5, section 30]. There is pure graph-theoretic work on operation digraphs[12, 7] as well as an algebraic theory of monounary algebras[9], which are the corresponding algebraic structures.

1.2. **Operation Matrices.** For the rest of this paper we take our operations to be defined on a finite set of inputs. Matrices can be used to encode all of the relevant information about a digraph. In order to do this we fix a canonical ordering on any underlying set we use.

Definition (Adjacency matrix)**.** Let $G(V,E)$ be a digraph, let $|V| = n$, and fix an order on the vertex set V. The *adjacency matrix* A for G under the given order on V is the $n \times n$ matrix whose ij-entry is 1 if there is an edge in G from v_i to v_j and 0 otherwise.

We can use adjacency matrices to study unary operations, as each operation digraph has a corresponding matrix. Below we give the adjacency matrices for the six operation digraphs depicted previously. We write A^L_{fs} to indicate the adjacency matrix of G^L_{fs} and similarly write A^R_{fs} to indicate the adjacency matrix of G^R_{fs}. Again we omit the superscript because the operations under consideration are commutative.

$$A_{+0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A_{+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad A_{+2} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$
$$A_{\times 0} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad A_{\times 1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A_{\times 2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

It is not difficult to see that in general any adjacency matrix for an operation digraph will have a single 1 in each row and that those corresponding to bijections will be permutation matrices. In this case $A_{+0} = A_{\times 1}$, since adding 0 and multiplying by 1 perform the same action on $\mathbb{Z}_3$. This is equivalent to noting that 0 and 1 are the identity elements for their respective binary operations or that as functions from $\mathbb{Z}_3$ to itself $f(x) \coloneqq 1x$ and $g(x) \coloneqq x + 0$ are the same.

We have a similar representation for the elements of the underlying set S. Let us identify the element s_i with the row vector whose i-entry is 1 and whose other entries are 0. Continuing our $\mathbb{Z}_3$ example, we have the following identifications.

$$s_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad s_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \quad s_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Remember that the entries in the vectors are elements of $\mathbb{C}$, while the left-hand-sides indicate members of $\mathbb{Z}_3$.

Multiplying a vector by the adjacency matrix of an operation digraph corresponds to applying the corresponding function to the corresponding element. That is, instead of computing $1 + 2 = 0$ in $\mathbb{Z}_3$ we can compute

$$s_2 A_{+1} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = s_0$$

or

$$s_1 A_{+2} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = s_0$$

There are two competing conventions here. Usually when we regard matrices as linear transformations we think of them as mapping column vectors on the right into row vectors. Graph theory indicates the opposite behavior, with function application occurring on the right.

1.3. **Graph Treks.** Recall that given a graph $G = (V, E)$, which need not be simple and may be directed, we have the following theorem.

Theorem. *Let A be the adjacency matrix for G with a given vertex ordering. Then $(A^k)_{ij}$ for $k \in \mathbb{N}$ is the number walks of length k from v_i to v_j in G.*

Now instead suppose we also have a graph H on the same set of vertices under the same ordering but with a possibly distinct set of edges from those in G. Let B be the adjacency matrix for H. Then it is natural to consider the significance of $(AB)_{ij}$ where AB is the usual matrix product of A and B. The following definition and theorem provide a useful way to interpret such an expression.

Definition. Let $(G_1, G_2, \ldots, G_k)$ be a tuple of graphs on a common set of vertices V. A *trek* (or (v_i, v_j)-*trek*) on $(G_1, G_2, \ldots, G_k)$ is an ordered list of vertices and edges $v_i, e_1, \ldots, e_k, v_j$ where $e_t \in E(G_t)$ is an edge joining the vertices before and after it in the list.

Theorem. *Let $(G_1, G_2, \ldots, G_k)$ be a tuple of graphs on a set of vertices V under a given vertex ordering and let $A_1, A_2, \ldots, A_k$ be the corresponding adjacency matrices. Then $(A_1 A_2 \cdots A_k)_{ij}$ is the number of treks on $(G_1, G_2, \ldots, G_k)$ of length k from v_i to v_j.*

Proof. Note that by definition the number of treks of length 1 from v_i to v_j along an edge from G_1 is given by $(A_1)_{ij}$. Now suppose inductively that we have that $(A_1 A_2 \cdots A_{k-1})_{ir}$ is the number of treks of length $k-1$ from v_i to v_r whose t^{th} step is along an edge from G_t.

Any trek of length k from v_i to v_j consists of a trek of length $k-1$ from v_i to v_r followed by a trek of length 1 (an edge) from v_r to v_j for some vertex $v_r \in V$. By our inductive hypothesis there are $(A_1 A_2 \cdots A_{k-1})_{ir}$ treks of the first kind and there are $(A_k)_{rj}$ of the second. For each $v_r \in V$ the number of treks of length k from v_i to v_j which pass through v_r on their penultimate step is

$$(A_1 A_2 \cdots A_{k-1})_{ir} (A_k)_{rj}.$$

The total number of all treks of length k from v_i to v_j is then the sum over all possible v_r of this quantity, so there are

$$\sum_r (A_1 A_2 \cdots A_{k-1})_{ir} (A_k)_{rj}$$

such treks, but this is precisely $(A_1 A_2 \cdots A_k)_{ij}$. □

This interpretation of such a product has obvious applications in finding walks in a graph G subject to a variety of secondary conditions by taking each of the G_t to be a subgraph of G. We now proceed to make use of the interpretation for a somewhat less literal purpose: counting solutions to equations in algebraic structures.

1.4. **Counting Solutions to Equations.** We now continue to examine $\mathbb{Z}_3$ by noting that if x is a solution to the equation $2x+1=0$ in $\mathbb{Z}_3$ then there must be a trek of length 2 from x to 0 whose first step is along an edge from $G_{\times 2}$ and whose second step is along an edge from G_{+1}. We can then check whether such an x exists by multiplying the corresponding adjacency matrices $A_{\times 2}$ and A_{+1}. We find that

$$A_{\times 2}A_{+1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

from which we conclude that there is exactly one such trek, which begins at $x=1$. The only solution to $2x+1=0$ in $\mathbb{Z}_3$ is then $x=1$. This process for solving equations can be stated in general as follows.

We first introduce a bit of notation. Let $\{f_p\}_{p\in P}$ be an indexed set of functions from S to itself and let Q be a finite (possibly empty) sequence consisting of elements of P. Also let $s \in S$. When Q is the empty sequence we write $f^Q(s)$ to indicate the element s itself. When Q is a nonempty sequence with a last element q, we write $f^Q(s)$ to denote $f_q(f^{Q^*}(s))$, where Q^* is the subsequence of Q which contains all but the last entry q in Q.

Theorem. *Let S be an ordered finite set of elements and let $\{f_p\}_{p\in P}$ where $f_p\colon S \to S$ be an indexed collection of functions. Let $G_p = G(S, E_p)$ be the operation digraph for f_p and let A_p be the adjacency matrix for G_p under the given ordering for S. If $Q = \{q_n\}_{n=1}^{k}$ is a finite sequence of k elements of P and $y = s_j$ is a fixed element of S we have that the number of $x \in S$ for which $f^Q(x) = y$ is exactly $\sum_{i=1}^{|S|}\left(\prod_{n=1}^{k}(A_{q_n})\right)_{ij}$.*

Proof. Since we take $G_p = G(S, E_p)$ to be an operation digraph, the edge set E_p is defined as $E_p = \{(s, f_p(s)) \mid s \in S\}$. Any edge $(u,v) \in E_p$ corresponds to obtaining v by applying f_p to u, so any such edge tells us that the pair $x=u, y=v$ is a solution to the equation $f_p(x) = y$. It follows that a finite trek from x to y along operation digraphs can be interpreted as a sequence of true equations indexed by some sequence $Q = \{q_1, \ldots, q_k\}$. That sequence of equations can be written as $\{f_{q_n}(u_n) = v_n\}_{n=1}^{k}$, where $v_n = u_{n+1}$. That is, the output of one function is the input of the next in this view of a valid trek.

We can then see that taking the first and last vertices in the sequence of u_n and v_n solving each of the successive equations $f_{s_n}(x) = y$ gives us the elements u and v respectively which solve an equation of the form $f_{q_k}(\ldots f_{q_2}(f_{q_1}(x))) = y$. We then have that each such trek corresponds to exactly one such pair solving the equation in question. As the matrix product $\prod_{n=1}^{k}(A_{q_n})$ has the number of such treks from s_i to s_j as its ij^{th} entry, the total number of entries in the j^{th} column which are nonzero gives the number of such treks beginning at any vertex and ending at s_j and hence the number solutions to the single-variable equation $f^Q(x) = y$ for any fixed $y \in S$, as well.

We also know that if $\left(\prod_{n=1}^{k}(A_{q_n})\right)_{ij} \neq 0$ then $\left(\prod_{n=1}^{k}(A_{q_n})\right)_{ij} = 1$. This is because the number of valid treks from s_i to s_j is a nonnegative integer and assuming that there exist two or more such treks from s_i to s_j leads us to conclude that at some step along the trek, say the n^{th} one, there are two distinct v_n such that $f_{q_n}(u_n) = v_n$, which contradicts that f_{q_n} is a function. It then follows that each entry in the matrix is either 0 or 1, with the sum of all the (nonzero) entries in a given column j giving the total number of valid treks beginning at any vertex s_i and ending at s_j, which is also the number of solutions $x = s_i$ to $f^Q(x) = y$ for a fixed $y = s_j$. We can take the total succinctly by summing over all rows i, so the number of solutions x is $\sum_{i=1}^{|S|} \left(\prod_{n=1}^{k}(A_{q_n})\right)_{ij}$. □

Note that the calculation in the example above actually gave us the adjacency matrix for the digraph corresponding to the map $x \mapsto 2x+1$ in $\mathbb{Z}_3$, from which we can obtain information about the solutions to equations of the form $2x+1 = y$. We expand on this idea in order to obtain lower bounds on the number of solutions to equations.

In order to obtain bounds on the number of distinct y for which there is a solution to an equation of the form $f^Q(x) = y$, we make use of a theorem due to Sylvester.

Theorem (Sylvester's rank inequality)**.** *Let U, V, and W be finite-dimensional vector spaces, let A be a linear transformation from V to W, and let B be a linear transformation from U to V. Then* $\operatorname{rank} AB \geq \operatorname{rank} A + \operatorname{rank} B - \dim(V)$, *where AB is the matrix product of the matrices corresponding to A and B.*

In particular, if A and B are linear transformations from a vector space V to itself then we have that both $\operatorname{rank} AB \geq \operatorname{rank} A + \operatorname{rank} B - \dim(V)$ and $\operatorname{rank} BA \geq \operatorname{rank} A + \operatorname{rank} B - \dim(V)$. Also, if we have a third linear transformation C from V to itself then we can conclude that

$$\begin{aligned}\operatorname{rank} ABC &\geq \operatorname{rank} AB + \operatorname{rank} C - \dim(V)\\ &\geq \operatorname{rank} A + \operatorname{rank} B + \operatorname{rank} C - 2\dim(V).\end{aligned}$$

By induction we see that for a finite collection of such transformations $\{A_i\}_{i\in I}$ we have $\operatorname{rank} \prod_{i\in I} A_i \geq \left(\sum_{i\in I} \operatorname{rank} A_i\right) - (|I|-1)\dim V$.

As the adjacency matrix for an operation digraph for an operation from a finite set S to itself can be viewed as a linear transformation from $\mathbb{C}^{|S|}$ to itself, we can apply that last statement to find a lower bound for the number of distinct y for which there exist solutions to an equation of the form $f^Q(x) = y$.

We will consider the equation

$$\left((3(x+2))^3\right)^{\left((3(x+2))^3\right)} = y$$

over $S = \mathbb{Z}_4$ as an example. Let $f_1(x) = x+2$, $f_2(x) = 3x$, $f_3(x) = x^3$, and $f_4(x) = x^x$. Note that the equation under consideration can be rewritten as $f^Q(x) = y$, where Q is the sequence $(1,2,3,4)$. Each of these functions has an associated operation digraph over $\mathbb{Z}_4$. The standard adjacency matrices corresponding to each function are as follows.

$$A_1 = A_{+2} = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 \end{bmatrix} \quad A_2 = A_{\times 3} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$A_3 = A^R_{\wedge 3} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_4 = A^R_{\uparrow 2} = \begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since $\operatorname{rank} A_{+2} = \operatorname{rank} A_{\times 3} = 4$ and $\operatorname{rank} A^R_{\wedge 3} = \operatorname{rank} A^R_{\uparrow 2} = 3$, we have that

$$\begin{aligned}\operatorname{rank} \prod_{n=1}^{4} A_n &\geq \left(\sum_{n=1}^{4} \operatorname{rank} A_n \right) - (|I| - 1)|S| \\ &= (4 + 4 + 3 + 3) - (4 - 1)4 \\ &= 2.\end{aligned}$$

It then must be that there are at least two nonzero columns in the resultant matrix and therefore at least two distinct $y \in \mathbb{Z}_4$ such that the equation in question has at least one solution $x \in \mathbb{Z}_4$.

We also have the following lemma, which tells us that we can apply the result of the previous section directly to functions on a finite set without resorting to the formalism invoked earlier.

Proposition (Sylvester's inequality for functions)**.** *Let X, Y, and Z be finite sets and let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions. Then*

$$|(g \circ f)(X)| \geq |f(X)| + |g(Y)| - |Y|.$$

Proof. Let $T_z = \{y \in Y \mid g(y) = z\}$ and let $T \subset Y$ such that $|T \cap T_z| = 1$ for all $z \in Z$. That is, let T be a set containing exactly one preimage under g of each element in $g(Y)$ and no other elements. Consider that for every $y \in f(X) \cap T$ we have an x where $f(x) = y$ such that $(g \circ f)(x) \in (g \circ f)(X)$. It follows that

$$|(g \circ f)(X)| \geq |f(X) \cap T|$$

with equality where T is chosen so that $f(x) \cap T$ is maximal. Since we know for any such intersection of sets that

$$|f(X) \cap T| \geq |f(X)| + |T| - |Y|$$

and we know that $|T| = |g(T)| = |g(Y)|$, it follows that

$$|f(X) \cap T| \geq |f(X)| + |g(Y)| - |Y|.$$

Thus, $|(g \circ f)(X)| \geq |f(X)| + |g(Y)| - |Y|$, as desired. □

One can see that this proposition is analogous to that of Sylvester, with the linear transformations and dimensions of the former corresponding to the functions and cardinalities of the latter. This lemma shows that our use of operation digraphs and linear algebra to obtain a lower bound on the number of solutions to equations of the form $f^S(x) = y$ was actually unnecessary. In this case we likely could have made the observation about functions directly, but the general method applied here may allow one to produce less obvious statements about functions by translating statements from linear algebra. Additionally, this analysis paved the way for the study of binary operations.

2. Binary Operations

We now repeat what we just did, but with binary operations instead. Again, our binary operations are of the from $f\colon S^2 \to S$ for some underlying set S.

2.1. **Operation Hypergraphs.** We can view a binary operation as a set

$$\{(s_i, s_j, f(s_i, s_j)) \mid s_i, s_j \in S\}.$$

This set can be seen as the edge set of a directed 3-uniform hypergraph[1].

Definition (Operation hypergraph)**.** Let $f\colon S^2 \to S$ be a binary operation. The *operation hypergraph* of f, written G_f, is given by $G_f = G(S, E)$ where

$$E = \{(s_i, s_j, f(s_i, s_j)) \mid s_i, s_j \in S\}.$$

For example, the operation hypergraph for $\mathbb{Z}_3$ under addition is

$$G_+ = \{(0,0,0),(0,1,1),(0,2,2),(1,0,1),(1,1,2),(1,2,0),(2,0,2),(2,1,0),(2,2,1)\}$$

and the operation hypergraph for $\mathbb{Z}_3$ under multiplication is

$$G_\times = \{(0,0,0),(0,1,0),(0,2,0),(1,0,0),(1,1,1),(1,2,2),(2,0,0),(2,1,2),(2,2,1)\}.$$

2.2. **Operation Tensors.** Tensors can be used to encode all the relevant information about a hypergraph. We use the naïve analog of the directed adjacency matrix here[4]. Other authors have explored different generalizations[13, 14].

Definition (Adjacency tensor)**.** Let $G(V, E)$ be a 3-uniform hypergraph, let $|V| = n$, and fix an order on the vertex set V. The *adjacency tensor* A for G under the given order on V is the $n \times n \times n$ hypermatrix whose ijk-entry is 1 if (v_i, v_j, v_k) is an edge in G and 0 otherwise.

Given a binary operation f we write A_f to indicate the adjacency tensor of the operation hypergraph of f. Recall that given such a tensor we can obtain a bilinear map $A_f\colon \mathbb{C}^S \times \mathbb{C}^S \to \mathbb{C}^S$ where given $x_1 = (a_s)_{s\in S}$ and $x_2 = (b_s)_{s\in S}$ from $\mathbb{R}^S$ we define

$$A_f(x_1, x_2) \coloneqq \sum_{s_i, s_j, s_k \in S} a_{s_i} b_{s_j} (A_f)_{ijk} s_k = \sum_{s_i, s_j \in S} a_{s_i} b_{s_j} f(s_i, s_j).$$

Since f maps basis elements of $\mathbb{C}^S$ to other basis elements of $\mathbb{C}^S$, we can always extend a binary operation f to a bilinear map in this way.

For example, given A_+ for the addition operation on $\mathbb{Z}_3$ and $x_1, x_2 \in \{0, 1, 2\}$ we have that

$$A_+(x_1, x_2) = \sum_{s_i, s_j \in \mathbb{Z}_3} a_{s_i} b_{s_j} (s_i +_{\mathbb{Z}_3} s_j) = x_1 +_{\mathbb{Z}_3} x_2,$$

so A_+ agrees with f where both are defined. It follows immediately from the definition that a binary operation f and this bilinear map always agree in this sense. In a slightly more exotic calculation, we can also "add $\frac{1}{2}x_1$ and x_2 in $\mathbb{Z}_3$" where $x_1, x_2 \in \{0, 1, 2\}$. We see that

$$A_+\left(\frac{1}{2}x_1, x_2\right) = \frac{1}{2} \sum_{s_i, s_j \in \mathbb{Z}_3} a_{s_i} b_{s_j} (s_i +_{\mathbb{Z}_3} s_j) = \frac{1}{2}(x_1 +_{\mathbb{Z}_3} x_2).$$

Since we saw that matrix multiplication corresponded to composition of unary operations, we are led to consider what sort of hypermatrix operation corresponds to composing binary operations.

2.3. **Hypergraph Odysseys.** There are many ways to compose binary operations. Let $f, g \colon S^2 \to S$. One possible composite function is given by

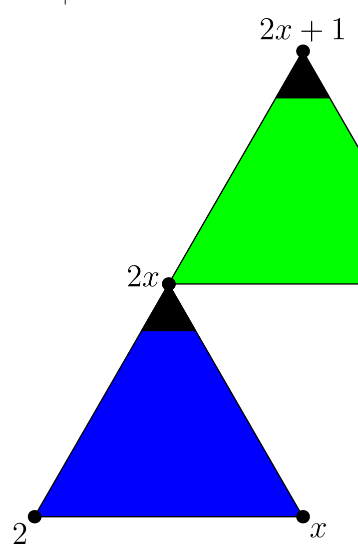
$$(x, y, z) \mapsto g(f(x, y), z)$$

while another is given by

$$(x, y, z) \mapsto f(f(x, x), g(x, f(x, f(y, z)))).$$

If we are going to handle binary operation composition in the same way that we handled unary operation composition we are going to need to define an infinite family of hypermatrix products, one for each possible way we can compose operations.

In order to do this, we refer to a generalized notion of the graph treks defined earlier. At that time we had considered the equation $2x + 1 = y$ in $\mathbb{Z}_3$, fixing y so that we had a single variable. We now let y vary and view the situation as follows, with the blue triangle representing an edge in $G_\times$ and the green triangle representing an edge in G_+.



In order for $2x + 1 = y$ to hold for a particular pair $(x, y) \in \mathbb{Z}_3^2$ we must have that there exists some $t \in \mathbb{Z}_3$ such that $(2, x, t)$ is an edge in $G_\times$ and $(t, 1, y)$ is an edge in G_+. Given such a t we have an example of a "generalized trek".

Let $(G_i = (S, E_i))_{i \in I}$ be a sequence of directed hypergraphs, each $\rho(i)$-uniform for some $\rho(i) \in \mathbb{N}$. The following definition is more general than we actually need for the moment, but the restriction to the case of only unary and binary operations is no easier to state. In the following definition we write $(\mu \circ \nu)(e)$ to indicate the result of applying ν to each of the entries of e which lie in the domain of ν, then applying μ to each of the entries of the tuple $\nu(e)$ so obtained which lie in the domain of μ.

Definition (μ, Σ-odyssey)**.** Let X and Y be sets of variables and take Σ to be a collection of pairs of the form (e, E) where $E = E_i$ for some $i \in I$ and $e \in (X \uplus Y)^{\rho(i)}$. If there exist evaluation maps $\mu \colon X \to S$ (the *endpoint evaluation map*) and $\nu \colon Y \to S$ (the *intermediate point evaluation map*) such that for each $(e, E) \in \Sigma$ we have that $(\mu \circ \nu)(e) \in E$ then we say that the collection of edges $\mathscr{O} = (\mu \circ \nu)(e)$ is a Σ-*odyssey* on the G_i. We say that X is the set of *end variables*,

Y is the set of *intermediate variables*, $\mu(X)$ is the set of *endpoints*, $\nu(Y)$ is the set of *intermediate points*, Σ is the *odyssey type*, and $|\Sigma|$ is the *length* of the odyssey. We call a Σ-odyssey $\mathscr{O}$ a μ, Σ*-odyssey* if $\mu\colon X \to S$ is the endpoint evaluation map of $\mathscr{O}$ for some fixed μ.

In our example above we have end variables $X = \{x, y, a, b\}$ and intermediate variable $Y = \{t\}$. Our Σ is given by $\Sigma = \{((a, x, t), G_\times), ((t, b, y), G_+)\}$. We consider only endpoint evaluation maps $\mu\colon X \to \mathbb{Z}_3$ such that $\mu(a) = 2$ and $\mu(b) = 1$. There are μ, Σ-odysseys for such μ. There are in fact 3. The first corresponds to $2(0) + 1 = 1$, with $\mu(x) = 0$, $\mu(y) = 1$, and $\nu(t) = 0$. The second corresponds to $2(1) + 1 = 0$, with $\mu(x) = 1$, $\mu(y) = 0$, and $\nu(t) = 2$. The last corresponds to $2(2) + 1 = 2$, with $\mu(x) = 2$, $\mu(y) = 2$, and $\nu(t) = 1$.

2.4. **Counting Solutions to Equations.** In the previous example we had to restrict ourselves to endpoint evaluation maps with certain properties in order to examine solutions to $2x + 1 = y$. It is more natural for us to define a generalized matrix product first in the context of equations with no constants, so we now consider the equation $ax + b = y$ over $\mathbb{Z}_3$. This equation has a corresponding product.

Let φ denote the logical formula

$$\varphi(a, b, x, y) \coloneqq (\exists t \in \mathbb{Z}_3)((a, x, t) \in G_\times \wedge (t, b, y) \in G_+).$$

This formula returns true if $ax + b = y$ and false otherwise. Equivalently, φ tells us whether or not there exists an odyssey as described previously. We can encode this logical formula as an arithmetic formula. Let A and B be arbitrary rank 3 tensors over $\mathbb{C}$. In an abuse of notation define a tensor φAB by

$$(\varphi AB)_{ijkl} \coloneqq \sum_{t \in \{0,1,2\}} A_{ikt} B_{tjl}.$$

The operation φ given by $(A, B) \mapsto \varphi AB$ is the *generalized matrix product* of A and B corresponding to the logical formula φ.

Since there is only one possible value for ax in $\mathbb{Z}_3$ we have that $(\varphi G_\times G_+)_{ijkl}$ is always either 0 or 1. In fact, by simple definition-chasing one finds that $\varphi G_\times G_+$ is the adjacency tensor for the composite operation

$$(a, b, x) \mapsto ax + b.$$

If we wish to restrict to the case where $a = 2$ and $b = 1$ we may simply obtain a new tensor by taking only those entries in φAB with the corresponding coordinates fixed. The resulting tensor has order 2 and is precisely the matrix product $A_{\times 2} A_{+1}$ obtained previously.

Although we refrain from presenting it here, these ideas lead to a tensor arithmetic that expresses concepts such as tensor contraction and some of the products used in spectral hypergraph theory[14] in a framework of relation composition.

3. Applications

Until this point we have primarily given definitions and results showing that those definitions are consistent with each other in some sense. In the section on unary operations we obtained bounds on the number of solutions to equations in one variable, but we could have given the argument without using our framework. In the second section we gave an equivalent way of counting the number of solutions to more general equations, but this is essentially nothing more than embedding function composition arithmetic in a larger space. We now present some applications of the perspective developed here.

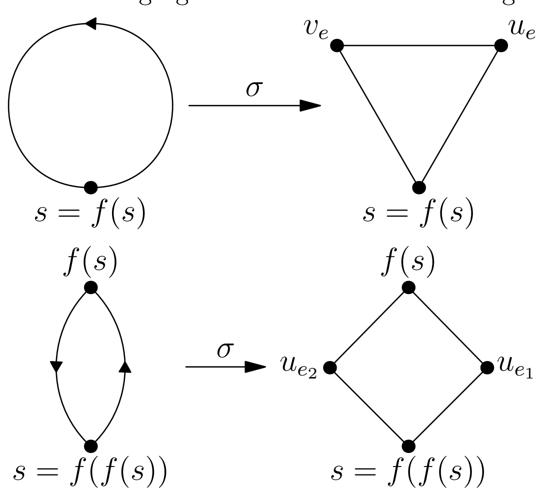
3.1. **Embedding Dimension.** We can study operations via the undirected versions of their operation hypergraphs. We begin with a unary operation.

Definition (Operation graph)**.** Let $f\colon S \to S$ be a unary operation. The *operation graph* of f, written $\bar{G}_f$, is the simple graph $G(V, E)$ which is constructed as follows. For each edge $e = (s, f(s))$ in G_f define

$$\sigma(e) := \begin{cases} \{(s, u_e), (u_e, v_e), (v_e, s)\} & \text{when } f(s) = s \\ \{(s, u_e), (u_e, f(s))\} & \text{when } f^2(s) = s \text{ and } f(s) \neq s \\ \{e\} & \text{otherwise} \end{cases}$$

where u_e and v_e are new vertices unique to the edge e. Take $E = \bigcup_{e \in E(G_f)} \sigma(e)$ and let V be the union of S and all the u_e and v_e generated by applying σ to edges $e \in E(G_f)$.

Note that this is not the graph of f in the more conventional sense, which we have been calling the operation digraph of f. We subdivide edges in order to pass to an undirected graph without ignoring loops ((s, s)) and 2-cycles ($(s, f(s))$ and $(f(s), s)$) in G_f. The following figure illustrates these two degeneracies.



Recall that every graph can be drawn without self-intersections in 3-dimensional Euclidean space, but some cannot be drawn without self-intersections in 2-dimensional Euclidean space. Those graphs which can be drawn in the plane are called *planar* and those graphs which cannot be drawn in the plane without self-intersections are called *nonplanar*.

Theorem. *Every operation graph is planar.*

Proof. A graph is planar if and only if it has a subgraph which is a subdivision of either K_5 or $K_{3,3}$, which are the complete graph on 5 vertices and the complete bipartite graph on 3 and 3 vertices, respectively.

Let $f\colon S \to S$ be a unary operation with operation graph $\bar{G}_f$. We show by contradiction that $\bar{G}_f$ cannot contain a subdivision of K_5 or $K_{3,3}$. Suppose first that $\bar{G}_f$ contains a subgraph H which is a subdivision of K_5. This subgraph contains five vertices, say s_1 through s_5, with a sequence of edges between any two. Note that the s_i for $i \in \{1, 2, 3, 4, 5\}$ must correspond to elements of S, since the dummy vertices u_e and v_e have degree 2 while the s_i have degree 4.

Consider the vertex s_1. Although $\bar{G}_f$ is undirected, each of the vertices adjacent to s_1 in H must come from subdividing a directed edge in G_f. Since G_f is an operation digraph, at most one of the vertices adjacent to s_1 in H may correspond to an outgoing edge in G_f.

Consider the sequence $s_1, s_2, \ldots, s_n$ of vertices along the path between s_1 and s_n in H (with dummy vertices omitted) where $n \in \{2, 3, 4, 5\}$. Suppose that $f(s_2) = s_1$ so that $\sigma((s_2, s_1))$ corresponds to an edge coming into s_1 in G_f. Since G_f is an operation digraph, it must be that $f(s_3) = s_2$, as otherwise we would have $f(s_2) = s_3$, contradicting that f is a function. Thus, every edge incident to s_1 in H which comes from an incoming edge in G_f must give us a corresponding edge incident to s_n in H which comes from an outgoing edge in G_f.

In a slight abuse of terminology, let the *in-degree* of s_i in H be the number of edges incident to s_i which come from incoming edges in G_f. We use the term *out-degree* here analogously. The preceding argument says that the total out-degree of the s_i is at least the sum of the in-degrees of the s_i. Since each s_i contributes at least 3 to this total and there are 5 such vertices, the total out-degree of the s_i must be at least 15. This is impossible, since there are only 5 such vertices, each of which may have out-degree at most 1. It must be that $\bar{G}_f$ cannot contain a subdivision of K_5, after all.

The argument that $\bar{G}_f$ cannot contain a subdivision of $K_{3,3}$ is essentially identical. We conclude that every $\bar{G}_f$ is planar. □

An alternative argument is that each of the points in S is either preperiodic under f or is part of a nonpreperiodic orbit (extending infinitely in one or both directions) of f. The graphs for the preperiodic points look like a cycles with directed trees leading into them and the graphs for the infinite orbits look like paths. All of these graphs are planar, so their disjoint union G_f is also planar, modulo cardinality issues if we look at functions on infinite sets.

The above reasoning was not wasted however, since we can actually make a slightly more general statement in this way.

Theorem. *Let H be a subdivision of a simple graph H' with n vertices, each of degree at least $k+1$ for $k \geq 2$. The graph H cannot appear as a subgraph of any operation graph if $k > \frac{n-1}{2}$.*

Proof. Every step in the previous proof carries through here, with H' taking the place of K_5 and $K_{3,3}$. We need $k \geq 2$ so that the vertices of H' cannot correspond to dummy vertices added when producing $\bar{G}_f$ from G_f, although the case where $k = 1$ is not very interesting anyway. □

Additionally, the proof presented has a chance of extending to hypergraphs, since it does not require us to understand the structure of the generalized orbits of a binary operation. We now give the magma analog of an operation graph.

Definition (Operation complex)**.** Let $f\colon S^2 \to S$ be a binary operation. The *operation complex* of f, written $\bar{G}_f$, is the simplicial complex whose 2-faces are the edges of the hypergraph $G(V, E)$, which is constructed as follows. Write $(a, b, c, d)_2$

to indicate the set of all 2-faces of the simplex with vertices a, b, c, and d. For each edge $e = (s_i, s_j, f(s_i, s_j))$ in G_f define

$$\sigma(e) \coloneqq \begin{cases} (s_i, u_e, v_e, w_e)_2 & \text{when } |\{s_i, s_j, f(s_i, s_j)\}| = 1 \\ (s_i, s_j, u_e, v_e)_2 & \text{when } |\{s_i, s_j, f(s_i, s_j)\}| = 2 \\ (s_i, s_j, s_k, u_e)_2 & \text{when } |\{s_i, s_j, f(s_i, s_j)\}| = 3 \text{ and } \tau e \in f \text{ for some} \\ & \text{nonidentity permutation } \tau \\ \{e\} & \text{otherwise} \end{cases}$$

where u_e, v_e, and w_e are new vertices unique to the edge e. Take $E = \bigcup_{e \in E(G_f)} \sigma(e)$ and let V be the union of S and all the u_e, v_e, and w_e generated by applying σ to edges $e \in E(G_f)$.

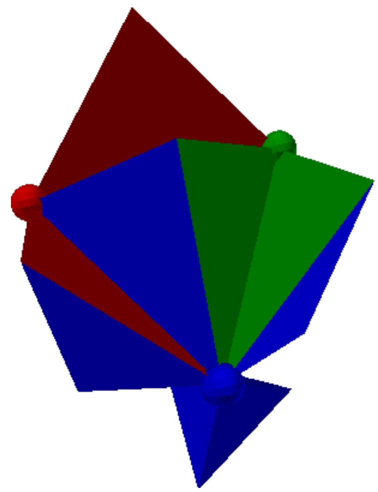
The above definition captures the "directed loops" and multiple edges possible in a 3-uniform directed hypergraph in the same way the operation graph of a unary operation did. A similar idea has been explored in the particular case of groups[8], where the extra structure provided by the group axioms made a more specialized construction possible.

It is known that every n-dimensional simplicial complex can be embedded without intersections into $\mathbb{R}^{2n+1}$[10]. Given any magma (S, f) we then know that $\bar{G}_f$ embeds into $\mathbb{R}^k$ but not $\mathbb{R}^{k-1}$ for some $k \in \{3, 4, 5\}$.

Definition (Embedding dimension)**.** Let (S, f) be a magma with operation complex $\bar{G}_f$. We refer to the minimal k such that the complex $\bar{G}_f$ embeds into $\mathbb{R}^k$ as the *embedding dimension* of the magma (S, f).

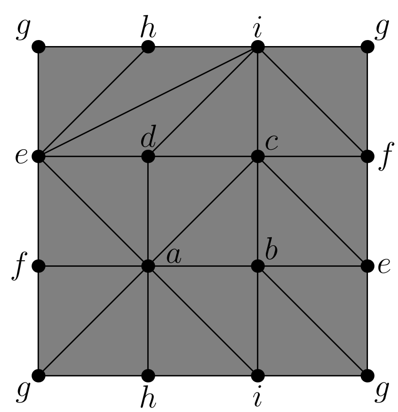
The situation here is more complex than for unary operations. First note that we can find examples of magmas of any finite order which embed into $\mathbb{R}^3$. To see this, let (S, f) be a magma such that for every $x, y \in S$, $x \neq y$, we have that either $f(x, y) = x$ or $f(x, y) = y$. Every edge $e \in G_f$ then contains at most 2 vertices which belong to S, with the others being dummy vertices. We claim that embedding $\bar{G}_f$ into $\mathbb{R}^3$ is accomplished by simply embedding the complete graph K_S whose vertices are the elements of S into $\mathbb{R}^3$. Suppose we have an embedding of K_S into $\mathbb{R}^3$ where each edge is mapped to a segment of a piecewise C_1 curve. We can fit the boundary of a tetrahedron into an envelope around the curve corresponding to any edge in K_S. We can take any such envelope to taper to the endpoints of the curve in question, so we can always prevent tetrahedra obtained from two different edges in G_f from overlapping. Finally, note that by the same reasoning we can place a tetrahedron in a neighborhood of any vertex of K_S, so the whole of $\bar{G}_f$ can be embedded into $\mathbb{R}^3$ without self-intersections.

There are also magmas of embedding dimension 3 without this property. Consider $(\mathbb{Z}_3, +)$. A brief inspection will reveal that this magma can be embedded in $\mathbb{R}_3$. Below is an image of such an embedding, with the elements of $\mathbb{Z}_3$ represented by spheres. The face coloring serves only to help distinguish faces.



Again let (S, f) be a magma. We demonstrate a technique for generating algebraic conditions which imply that the embedding dimension of (S, f) is at least 4. Recall that the Klein bottle cannot be embedded in $\mathbb{R}^3$ without self-intersection. We know the minimal triangulations of the Klein bottle[15] so we can orient such a triangulation to obtain a minimal algebraic rule which implies that a given magma has embedding dimension at least 4.

Consider the triangulation Kh12 from [15], which is pictured below. The horizontal edges are to be identified in parallel and the vertical edges in antiparallel.



We orient the faces of Kh12 in a manner consistent with the following partial operation table. Note that not every possible orientation can come from an operation. On each triangular face we may choose a left input, right input, and output. If we choose poorly we can designate two outputs for the same input pair in the same order.

	a	b	c	d	e	f	g	h	i
a	$\cdot$	c	d	e	f	g	h	i	b
b	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
c	$\cdot$	$\cdot$	$\cdot$	i	b	$\cdot$	$\cdot$	$\cdot$	$\cdot$
d	$\cdot$	$\cdot$	$\cdot$	$\cdot$	i	$\cdot$	$\cdot$	$\cdot$	$\cdot$
e	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	c	b	$\cdot$	$\cdot$
f	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	i	$\cdot$	c
g	$\cdot$	$\cdot$	$\cdot$	$\cdot$	h	$\cdot$	$\cdot$	$\cdot$	b
h	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	e
i	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

This "forbidden substructure" cannot appear in any magma with embedding dimension 3. We can extend our earlier example of magmas with embedding dimension 3 to produce a magma with embedding dimension 4. For each pair x, y for which $\cdot$ appears in the table above define $f(x, y) = x$. None of these new degenerate faces will change the embedding dimension of the magma, so the resulting operation has embedding dimension 4.

It is immediate that embedding dimension can only decrease when considering a submagma of a given magma. What relationship does embedding dimension have with taking homomorphic images and products of magmas? If it only goes down then we know that "magmas of embedding dimension at most k" is a variety and hence an equational class by Birkhoff's Theorem[2]. This would tell us that there is a set of identities which characterize such magmas (and hence their operation complexes). If not, we can show that it is impossible to produce such a characterization.

3.2. **Spectrum Calculation.** There is a very direct relationship between the spectrum of an operation digraph and the dynamics of the original function.

Theorem. *Let $f\colon S \to S$ be a function on a set S of size n. Let $m(j)$ denote the number of j-cycles under f and let Z_j denote the multiset which consists of $m(j)$ copies of each j^{th} root of unity. The nonzero part of the spectrum of A_f is the multiset union $\bigcup_j Z_j$.*

Proof. Fix an order on S which places the k periodic points of S under f first, followed by the nonperiodic points of S. With respect to this order on S we can write the adjacency matrix A_f in block form as

$$A_f = \begin{bmatrix} \mathbb{S} & \mathbb{O} \\ \mathbb{T} & \mathbb{U} \end{bmatrix}$$

where $\mathbb{S}$ is a $k \times k$ permutation matrix and $\mathbb{U}$ is an $(n-k) \times (n-k)$ matrix. We know that $\mathbb{O}$ is an all-zero matrix because every periodic point is mapped to another periodic point.

We can take our order on S so that $\mathbb{U}$ is a lower triangular matrix. To see this, note that every nonperiodic point in S is either mapped to a periodic point under f, in which case the corresponding row in $\mathbb{U}$ is all zero, or mapped to a nonperiodic point. We can define a partial order $\leq$ on the nonperiodic points of S where $s_i \leq s_j$ if $f(s_j) = s_i$. Put a total order on the nonperiodic elements of S so that s_i appears before s_j whenever $s_i \leq s_j$. This will place all of the 1 entries in $\mathbb{U}$ to the left of the main diagonal, since each nonperiodic point is then mapped to a point that precedes it in the order on S.

Let I_t denote the $t \times t$ identity matrix. Since A_f is lower block triangular we have that $\lambda I_n - A_f$ is also lower block triangular, whence $\det(\lambda I_n - A_f) = (\det(\lambda I_k - \mathbb{S}))(\det(\lambda I_{n-k} - \mathbb{U}))$. Thus, the spectrum of A_f is the multiset union of the spectra of $\mathbb{S}$ and $\mathbb{U}$. Note that the diagonal entries of $\mathbb{U}$ are all zero since no nonperiodic point may be mapped to itself. This implies that the spectrum of $\mathbb{U}$ is all zero. Thus, the entire nonzero spectrum of A_f is that of $\mathbb{S}$. Since $\mathbb{S}$ is a permutation matrix we have that $\mathbb{S}$ is a matrix of finite order. By an earlier analysis of the spectra of matrices of finite order[11], we have that the spectrum of $\mathbb{S}$ is precisely the previously asserted nonzero spectrum of A_f. □

In contrast with this complete description of the spectrum of an operation digraph, no such generic description of a spectrum for a uniform hypergraph is known to this author. A possible future project is to use the special case of operation hypergraphs as a stepping stone to the general case.

References

[1] Claude Berge. *Hypergraphs: Combinatorics of Finite Sets*. Elsevier Science Publishers B.V., 1989. ISBN: 0444874895 (cit. on p. 8).

[2] Clifford Bergman. *Universal Algebra: Fundamentals and Selected Topics*. Chapman and Hall/CRC, 2011. ISBN: 978-1-4398-5129-6 (cit. on p. 15).

[3] J. R. Doyle. "Preperiodic points for quadratic polynomials with small cycles over quadratic fields". In: *ArXiv e-prints* (Sept. 2015). arXiv: 1509.07098 [math.NT] (cit. on p. 2).

[4] Joel Friedman and Avi Wigderson. *On the Second Eigenvalue of Hypergraphs*. 1989 (cit. on p. 8).

[5] Joseph A. Gallian. *Contemporary Abstract Algebra*. 7th. Brooks/Cole, 2010. USA (cit. on p. 2).

[6] Olexandr Ganyushkin and Volodymyr Mazorchuk. *Classical Finite Transformation Semigroups: An Introduction*. First. Algebra and Applications 9. Springer-Verlag London, 2009. ISBN: 9781848002807, 9781848002814, 1848002807 (cit. on p. 2).

[7] Frank Harary and Ron Read. "The probability of a given 1-choice structure". In: *Psychometrika* 31.2 (1966), pp. 271–278. ISSN: 1860-0980. DOI: 10.1007/BF02289514. URL: http://dx.doi.org/10.1007/BF02289514 (cit. on p. 3).

[8] M. Herman, J. Pakianathan, and E. Yalcin. "On a canonical construction of tesselated surfaces via finite group theory, Part I". In: *ArXiv e-prints* (Oct. 2013). arXiv: 1310.3848 [math.GT] (cit. on p. 13).

[9] Gábor Horváth et al. "The number of monounary algebras". In: *Algebra universalis* 66.1 (2011), p. 81. ISSN: 1420-8911. DOI: 10.1007/s00012-011-0147-y. URL: http://dx.doi.org/10.1007/s00012-011-0147-y (cit. on p. 3).

[10] G.M. Ziegler Jiri Matousek A. Björner. *Using the Borsuk-Ulam theorem: lectures on topological methods in combinatorics and geometry*. Universitext. Springer, 2003. ISBN: 3540003622,9783540003625 (cit. on p. 13).

[11] Reginald Koo. "A Classification of Matrices of finite Order over $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{Q}$". In: *Mathematics Magazine* 76.2 (Apr. 2003), pp. 143–148 (cit. on p. 16).

[12] James B. Orlin. *Line-Digraphs, Arborescences, and Theorems of Tutte and Knuth*. 1978 (cit. on p. 3).

[13] Kelly J. Pearson. "Eigenvalues of the Adjacency Tensor on Products of Hypergraphs". In: *Int. J. Contemp. Math. Sciences* 8.4 (2013), pp. 151–158 (cit. on p. 8).

[14] Kelly J. Pearson and Tan Zhang. "On Spectral Hypergraph Theory of the Adjacency Tensor". In: *Graphs and Combinatorics* 30.5 (2014), pp. 1233–1248. ISSN: 1435-5914. DOI: 10.1007/s00373-013-1340-x. URL: http://dx.doi.org/10.1007/s00373-013-1340-x (cit. on pp. 8, 10).

[15] T. Sulanke. "Note on the Irreducible Triangulations of the Klein Bottle". In: *ArXiv Mathematics e-prints* (July 2004). eprint: math/0407008 (cit. on p. 14).

[16] Douglas B. West. *Introduction to Graph Theory*. Second. Pearson, 2002 (cit. on p. 2).