# MULTIPLAYER ROCK-PAPER-SCISSORS

CHARLOTTE ATEN

## Contents



## 1. Introduction

The game of Rock-Paper-Scissors (RPS) involves two players simultaneously choosing either rock ($r$), paper ($p$), or scissors ($s$). Informally, the rules of the game are that "rock beats scissors, paper beats rock, and scissors beats paper". That is, if one player selects rock and the other selects paper then the latter player wins, and so on. If two players choose the same item then the round is a tie.

A *magma* is an algebra $\mathbf{A} := (A, f)$ consisting of a set $A$ and a single binary operation $f\colon A^2 \to A$. We will view the game of RPS as a magma. We let $A := \{r, p, s\}$ and define a binary operation $f\colon A^2 \to A$ where $f(x, y)$ is the winning item among $\{x, y\}$. This operation is given by the table below and completely describes the rules of RPS. In order to play the first player selects a member of $A$, say $x$, at the same time that the second player selects a member of $A$, say $y$. Each player who selected $f(x, y)$ is the winner. Note that it is possible for both players to win, in which case we have a tie.

| | $r$ | $p$ | $s$ |
|---|---|---|---|
| $r$ | $r$ | $p$ | $r$ |
| $p$ | $p$ | $p$ | $s$ |
| $s$ | $r$ | $s$ | $s$ |

In general we have a class of *selection games*, which are games consisting of a collection of items $A$, from which a fixed number of players $n$ each choose one, resulting in a tuple $a \in A^n$, following which the round's winners are those who chose $f(a)$ for some fixed rule $f\colon A^n \to A$. We refer to an algebra $\mathbf{A} := (A, f)$ with a single basic $n$-ary operation $f\colon A^n \to A$ as an *$n$-ary magma* or an *$n$-magma*. We will sometimes abuse this terminology and refer to an $n$-ary magma $\mathbf{A}$ simply as a *magma*. Each such game can be viewed as an $n$-ary magma and each $n$-ary magma

*Date*: 2018.

email: caten2@u.rochester.edu

Thanks to Scott Kirila for pointing out the result we use in section 2.
This research was supported in part by the people of the Yosemite Valley.

can be viewed as a game in the same manner, providing we allow for games where we keep track of who is "player 1", who is "player 2", etc. Again note that any subset of the collection of players might win a given round, so there can be multiple player ties.

The classic RPS game has several desirable properties. Namely, RPS is, in terms we proceed to define,

(1) conservative,
(2) essentially polyadic,
(3) strongly fair, and
(4) nondegenerate.

Let $\mathbf{A} := (A, f)$ be an $n$-magma. We say that an operation $f\colon A^n \to A$ is *conservative* when for any $a_1, \dots, a_n \in A$ we have that $f(a_1, \dots, a_n) \in \{a_1, \dots, a_n\}$[1, p.94]. Similarly we call $\mathbf{A}$ *conservative* when $f$ is conservative. We say that an operation $f\colon A^n \to A$ is *essentially polyadic* when there exists some $g\colon \mathrm{Sb}(A) \to A$ such that for any $a_1, \dots, a_n \in A$ we have $f(a_1, \dots, a_n) = g(\{a_1, \dots, a_n\})$. Similarly we call $\mathbf{A}$ *essentially polyadic* when $f$ is essentially polyadic. We say that $f$ is *fair* when for all $a, b \in A$ we have $\left|f^{-1}(a)\right| = \left|f^{-1}(b)\right|$. Let $A_k$ denote the members of $A^n$ which have $k$ distinct components for some $k \in \mathbb{N}$. We say that $f$ is *strongly fair* when for all $a, b \in A$ and all $k \in \mathbb{N}$ we have $\left|f^{-1}(a) \cap A_k\right| = \left|f^{-1}(b) \cap A_k\right|$. Similarly we call $\mathbf{A}$ *(strongly) fair* when $f$ is (strongly) fair. Note that if $f$ (respectively, $\mathbf{A}$) is strongly fair then $f$ (respectively, $\mathbf{A}$) is fair, but the reverse implication does not hold. We say that $f$ is *nondegenerate* when $|A| > n$. Similarly we call $\mathbf{A}$ *nondegenerate* when $f$ is nondegenerate.

Thinking in terms of selection games we say that $\mathbf{A}$ is conservative when each round has at least one winning player. We say that $\mathbf{A}$ is essentially polyadic when a round's winning item is determined solely by which items were played, not taking into account which player played which item or how many players chose a particular item. We say that $\mathbf{A}$ is fair when each item has the same probability of being the winning item (or tying). We say that $\mathbf{A}$ is strongly fair when each item has the same chance of being the winning item when exactly $k$ distinct items are chosen for any $k \in \mathbb{N}$. Note that this is not the same as saying that each player has the same chance of choosing the winning item (respectively, when exactly $k$ distinct items are chosen). When $\mathbf{A}$ is *degenerate* (i.e. not nondegenerate) we have that $|A| \leq n$. In the case that $|A| \leq n$ we have that all members of $A_{|A|}$ have the same set of components. If $\mathbf{A}$ is essentially polyadic with $|A| \leq n$ it is impossible for $\mathbf{A}$ to be strongly fair unless $|A| = 1$.

Extensions of RPS which allow players to choose from more than the three eponymous items are attested historically. The French variant of RPS gives a pair of players 4 items to choose among[6, p.140]. In addition to the usual rock, paper, and scissors there is also the well ($w$). The well beats rock and scissors but loses to paper. The corresponding Cayley table is given below. This game is not fair, as $\left|f^{-1}(r)\right| = 3$ yet $\left|f^{-1}(p)\right| = 5$. It is nondegenerate since there are 4 items for 2 players to chose among. It is also conservative and essentially polyadic.

| | $r$ | $p$ | $s$ | $w$ |
|---|---|---|---|---|
| $r$ | $r$ | $p$ | $r$ | $w$ |
| $p$ | $p$ | $p$ | $s$ | $p$ |
| $s$ | $r$ | $s$ | $s$ | $w$ |
| $w$ | $w$ | $p$ | $w$ | $w$ |

There has been some recent recreational interest in RPS variants with larger numbers of items from which two players may choose. For example, the game Rock-Paper-Scissors-Spock-Lizard[3] (RPSSL) is attested in the popular culture. The Cayley table for this game is given below, with $v$ representing Spock and $l$ representing lizard. This game is conservative, essentially polyadic, strongly fair, and nondegenerate.

| | $r$ | $p$ | $s$ | $v$ | $l$ |
|---|---|---|---|---|---|
| $r$ | $r$ | $p$ | $r$ | $v$ | $r$ |
| $p$ | $p$ | $p$ | $s$ | $p$ | $l$ |
| $s$ | $r$ | $s$ | $s$ | $v$ | $s$ |
| $v$ | $v$ | $p$ | $v$ | $v$ | $l$ |
| $l$ | $r$ | $l$ | $s$ | $l$ | $l$ |

It is folklore that the only "valid" RPS variants for two players use an odd number of items. Currently this is mentioned on the Wikipedia entry for Rock-Paper-Scissors without citation[5] and with a reference to a collection of such games[4]. In our language we have the following result. We give a proof of a more general statement in the next section.

**Theorem.** *Let* $\mathbf{A}$ *be a selection game with* $n = 2$ *which is essentially polyadic, strongly fair, and nondegenerate and let* $m := |A|$. *We have that* $m \neq 1$ *is odd. Conversely, for each odd* $m \neq 1$ *there exists such a selection game.*

In the present paper we explore selection games for more than 2 simultaneous players. We give a numerical constraint on which $n$-magmas of order $m$ can be essentially polyadic, strongly fair, and nondegenerate. We use this constraint to examine permissible values of $m$ for a fixed $n$ and vice versa. We describe some elementary algebraic properties of magmas which are generalized RPS games.

## 2. RPS Magmas

The magmas we are interested in are those corresponding to selection games which have the four desirable properties possessed by Rock-Paper-Scissors.

**Definition** (RPS magma)**.** Let $\mathbf{A} := (A, f)$ be an $n$-ary magma. When $\mathbf{A}$ is conservative, essentially polyadic, strongly fair, and nondegenerate we say that $\mathbf{A}$ is an RPS *magma*. When $\mathbf{A}$ is an $n$-magma of order $m$ with these properties we say that $\mathbf{A}$ is an $\mathrm{RPS}(m, n)$ *magma*. We also use RPS and $\mathrm{RPS}(m, n)$ to indicate the classes of such magmas.

Note that in order for $\mathbf{A}$ to be fair we need that the number of items ($m$) divides the number of tuples $|A^n| = m^n$, which is always the case. This is certainly not a sufficient condition, as we have seen, for example, that the French variant of RPS is unfair.

Our first theorem generalizes directly to selection games with more than 2 players.

**Theorem.** *Let* $\mathbf{A}$ *be a selection game with* $n$ *players and* $m$ *items which is essentially polyadic, strongly fair, and nondegenerate. For all primes* $p \leq n$ *we have that* $p \nmid m$. *Conversely, for each pair* $(m, n)$ *with* $m \neq 1$ *such that for all primes* $p \leq n$ *we have that* $p \nmid m$ *there exists such a selection game.*

*Proof.* Since $\mathbf{A}$ is nondegenerate we must have that $m > n$.

Since $\mathbf{A}$ is strongly fair we must have that $\left|f^{-1}(a) \cap A_k\right| = \left|f^{-1}(b) \cap A_k\right|$ for all $k \in \mathbb{N}$. As the $m$ distinct sets $f^{-1}(a) \cap A_k$ for $a \in A$ partition $A_k$ and are all the same size we require that $m \mid |A_k|$. When $k > n$ we have that $A_k = \varnothing$ and obtain no constraint on $m$.

When $k \leq n$ we have that $A_k$ is nonempty. As we take $\mathbf{A}$ to be essentially polyadic we have that $f(x) = f(y)$ for all $x, y \in A_k$ such that $\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\}$. Let $B_k$ denote the collection of unordered sets of $k$ distinct elements of $A$. Note that the size of the collection of all members $x \in B_k$ such that $\{x_1, \ldots, x_n\} = \{z_1, \ldots, z_k\}$ for distinct $z_i \in A$ does not depend on the choice of distinct $z_i$. This implies that for a fixed $k \leq n$ each of the $m$ items must be the winner among the same number of unordered sets of $k$ distinct elements in $A$. We have that $|B_k| = \binom{m}{k}$ so we require that $m \mid |B_k| = \binom{m}{k}$ for all $k \leq n$.

Let

$$d(m, n) \coloneqq \gcd\left(\left\{ \binom{m}{k} \;\middle|\; 1 \leq k \leq n \right\}\right).$$

Since $m \mid \binom{m}{k}$ for all $k \leq n$ we must have that $m \mid d(m, n)$. Joris, Oestreicher, and Steinig showed that when $m > n$ we have

$$d(m, n) = \frac{m}{\operatorname{lcm}(\{\, k^{\varepsilon_k(m)} \mid 1 \leq k \leq n \,\})}$$

where $\varepsilon_k(m) = 1$ when $k \mid m$ and $\varepsilon_k(m) = 0$ otherwise[2, p.103]. Since we have that $m \mid d(m, n)$ and $d(m, n) \mid m$ it must be that $m = d(m, n)$ and hence

$$\operatorname{lcm}\left(\left\{ k^{\varepsilon_k(m)} \;\middle|\; 1 \leq k \leq n \right\}\right) = 1.$$

This implies that $\varepsilon_k(m) = 0$ for all $2 \leq k \leq n$. That is, no $k$ between 2 and $n$ inclusive divides $m$. This is equivalent to having that no prime $p \leq n$ divides $m$, as desired.

It remains to show that such games $\mathbf{A}$ exist when $m \neq 1$ and for all primes $p \leq n$ we have that $p \nmid m$. By this assumption we have that $k \nmid m$ whenever $2 \leq k \leq n$. Since

$$\binom{m}{k} = \frac{m!}{(m-k)!k!} = m \frac{(m-1) \cdots (m-k+1)}{k(k-1) \cdots (2)}$$

and none of the factors of $k!$ divide $m$ it must be that $m \mid \binom{m}{k}$ for each $2 \leq k \leq n$. This implies that $m \mid |B_k|$ for each $k \leq n$ so for each $k \leq n$ we can partition $B_k$ into $m$ subcollections $C_k \coloneqq \{C_{k,r}\}_{r \in A}$ indexed on the $m$ elements of $A$, each with $|C_{k,r}| = \frac{\binom{m}{k}}{m}$. With respect to this collection of partitions $C \coloneqq \{C_k\}_{1 \leq k \leq n}$ we define an $n$-ary operation $f \colon A^n \to A$ by $f(a_1, \ldots, a_n) \coloneqq r$ when $\{a_1, \ldots, a_n\} \in C_{k,r}$ for some $k \in \{1, \ldots, n\}$. This map is well-defined since each $\{a_1, \ldots, a_n\}$ contains exactly $k$ distinct elements for some $k \in \{1, \ldots, n\}$ and thus belongs to a unique member of one of the partitions $C_k$. In order to see that the resulting magma $\mathbf{A} \coloneqq (A, f)$ is essentially polyadic choose some $a_0 \in A$ and let $g \colon \operatorname{Sb}(A) \to A$ be given by

$$g(U) \coloneqq \begin{cases} r & \text{when } (\exists k) U \in C_{k,r} \\ a_0 & \text{when } |U| > n \end{cases}.$$

By construction we have that $f(a_1, \ldots, a_n) = g(\{a_1, \ldots, a_n\})$ for all $a_1, \ldots, a_n \in A$. The choice of $a_0$ was immaterial. We now show that $\mathbf{A}$ is strongly fair. Given

$r \in A$ we have that $f(a_1, \ldots, a_n) = r$ with $(a_1, \ldots, a_n) \in A_k$ when $\{a_1, \ldots, a_n\} \in C_{k,r}$. Note that the number of members of $A_k$ whose coordinates form the set $\{a_1, \ldots, a_n\}$ is the same as the number of members of $A_k$ whose coordinates form the set $\{b_1, \ldots, b_n\}$ for some other $(b_1, \ldots, b_n) \in A_k$. This implies that each of the $\left|f^{-1}(r) \cap A_k\right|$ have the same size for a fixed $k$ and hence $\mathbf{A}$ is strongly fair. To see that $\mathbf{A}$ is nondegenerate observe that if $1 \neq m < n$ then there is some prime $p$ dividing $m$. Since $p \leq m < n$ is prime we require that $p \nmid m$, a contradiction. We see that an essentially polyadic, strongly fair, nondegenerate $n$-ary magma always exists when $m \neq 1$ and for all primes $p \leq n$ we have that $p \nmid m$. □

We have given a description of all possible essentially polyadic, strongly fair, nondegenerate $n$-magmas. There is always at least one RPS magma for every $n \geq 2$, although for brevity we refrain from demonstrating this.

## 3. Items as a Function of Players and Vice Versa

Our numerical condition on the existence of an $\mathrm{RPS}(m,n)$ magma allows us to analyze how many items $m$ can be used by a fixed number $n$ of players in such a game. For the $n = 2$ case we see that the only prime $p \leq 2$ is 2 so $2 \nmid m$. For $n = 3$ we find that $2 \nmid m$ and $3 \nmid m$. As $2 \nmid m$ we have that $m \pmod 6 \in \{1, 3, 5\}$. As $3 \nmid m$ we have that $m \pmod 6 \in \{1, 2, 4, 5\}$. Combining these conditions we see that $\mathrm{RPS}(m,3)$ algebras can only exist for $m \equiv 1 \pmod 6$ or $m \equiv 5 \pmod 6$. Our example of an RPS 3-magma of order 5 was the smallest possible and we see that the next largest RPS 3-magmas have order $m = 6 + 1 = 7$. A similar analysis can be used to obtain a constraint on $m$ modulo the product of all primes $p \leq n$ for any fixed $n$. Since 2 and 3 are also the only primes $p \leq 4$ we find that $\mathrm{RPS}(m,4)$ magmas can only exist for $m \equiv 1 \pmod 6$ or $m \equiv 5 \pmod 6$. In the case of $n = 5$, however, we obtain that $m \pmod{30} \in \{1, 7, 11, 13, 17, 19, 23, 29\}$. The smallest RPS 5-magma thus has order 7.

Our numerical condition also allows us to fix the number of items $m$ and ask how many players $n$ may use that number of items. By our previous work this question is answered immediately.

**Theorem.** *Given a fixed $m$ there exists an $\mathrm{RPS}(m,n)$ magma if and only if $n < t(m)$ where $t(m)$ is the least prime dividing $m$.*

*Proof.* Suppose that there exists such a magma. We know that in this case $p \nmid m$ for all $p \leq n$. Since $t(m)$ is a prime dividing $m$ we must have that $n < t(m)$. Conversely, if $n < t(m)$ then all primes $p \leq n$ are less than $t(m)$ and as such do not divide $m$. It is possible to show that $\mathrm{RPS}(m,n)$ magmas always exist when this condition is met. □

Our result implies that the only $\mathrm{RPS}(2,n)$ magma has $n = 1$. Up to isomorphism this is the magma $(A, f)$ with $A = \{a, b\}$ and $f \colon A \to A$ the identity map. There are $\mathrm{RPS}(3,n)$ magmas for $n \leq 2$. The game of RPS is one such $\mathrm{RPS}(3,2)$ magma. There are $\mathrm{RPS}(4,n)$ magmas only for $n = 1$. Up to isomorphism this is again a set $A$ with $|A| = 4$ and $f = \mathrm{id}_A$. There are $\mathrm{RPS}(5,n)$ magmas for $n \leq 4$. The game of RPSSL is one such $\mathrm{RPS}(5,2)$ magma. In the full version of this paper we will use a $\mathbb{Z}_5$ action to obtain an example of a $\mathrm{RPS}(5,3)$ magma. Since $2 \mid 6$ the only $\mathrm{RPS}(6,n)$ magmas are unary. We can perform a similar analysis for any fixed number of items $m$.

## 4. Algebraic Properties of RPS Magmas

We give some basic algebraic properties of RPS magmas. Note that both the original RPS magma and the magma for the French variant are contained in the magma for RPSSL so subalgebras of RPS magmas may or may not be RPS magmas. Observe that any subset of the universe of a conservative magma is a subuniverse. For example, $\{r, s, l\}$ is a subuniverse for RPSSL and the corresponding subalgebra satisfies the numerical condition necessary for RPS magmas (being binary and of order 3), yet the corresponding subalgebra fails to be strongly fair.

The class of RPS magmas is as far from being closed under products as possible.

**Theorem.** *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be nontrivial* RPS $n$*-magmas with* $n > 1$*. The magma* $\mathbf{A} \times \mathbf{B}$ *is not an* RPS *magma.*

*Proof.* We show that $\mathbf{A} \times \mathbf{B}$ cannot be conservative. Let $\mathbf{A} \coloneqq (A, f)$ and let $\mathbf{B} \coloneqq (B, g)$. Let $x_1, \ldots, x_n \in A$ be distinct and let $y_1, \ldots, y_n \in B$ be distinct. Since $f$ and $g$ are conservative we have that $f(x) = x_i$ and $g(y) = y_j$ for some $i$ and $j$. It follows that $(f \times g)((x_1, y_1), \ldots, (x_n, y_n)) = (x_i, y_j)$. Either $i \neq j$, in which case $(x_i, y_j)$ is not one of the $(x_i, y_i)$ and $\mathbf{A} \times \mathbf{B}$ is not conservative, or $i = j$. In this latter case we have that $f$ is essentially polyadic so $f(\sigma(x)) = f(x)$ for any permutation $\sigma$ of the $x_i$. This implies that

$$f(x_2, \ldots, x_n, x_1) = f(x_1, \ldots, x_n) = x_i$$

so

$$(f \times g)((x_2, y_1), \ldots, (x_n, y_{n-1}), (x_1, y_n)) = (x_i, y_i).$$

Now $(x_i, y_i)$ does not appear among the arguments of $f \times g$ so again we see that $\mathbf{A} \times \mathbf{B}$ cannot be conservative and hence cannot be an RPS magma. □

## References


[1] Clifford Bergman. *Universal Algebra: Fundamentals and Selected Topics*. Chapman and Hall/CRC, 2011. ISBN: 978-1-4398-5129-6 (cit. on p. 2).

[2] H Joris, C Oestreicher, and J Steinig. "The greatest common divisor of certain sets of binomial coefficients". In: *Journal of Number Theory* 21.1 (1985), pp. 101–119. ISSN: 0022-314X. DOI: https://doi.org/10.1016/0022-314X(85)90013-7 (cit. on p. 4).

[3] *Rock Paper Scissors Spock Lizard*. Jan. 16, 2018. URL: http://www.samkass.com/theories/RPSSL.html (cit. on p. 3).

[4] *Rock Paper Scissors Variants*. Jan. 16, 2018. URL: http://www.umop.com/rps.htm (cit. on p. 3).

[5] *Rock-paper-scissors*. Jan. 16, 2018. URL: https://en.wikipedia.org/wiki/Rock%C3%A2%C2%80%C2%93paper%C3%A2%C2%80%C2%93scissors (cit. on p. 3).

[6] Gisèle Umbhauer. *Game Theory and Exercises*. 1st ed. Routledge Advanced Texts in Economics and Finance. Routledge, 2016. ISBN: 0415604214,978-0-415-60421-5,978-0-415-60422-2,978-1-315-66906-9 (cit. on p. 2).