Categorical models of linear logic

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Introduction

- Linear logic
- de Paiva's "Girard construction"
- Application to Petri nets

Introduction

- In this talk I will introduce linear logic, a resource-aware logic which generalizes classical logic.
- I will describe de Paiva's categorical model of classical linear logic.
- Finally, given time, I will mention how a similar construction allows one to model Petri nets.

Linear logic

- In classical logic we have the proof-rules weakening and contraction, given below.

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ \text{Weakening}_L \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ \text{Weakening}_R$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ \text{Contraction}_L \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ \text{Contraction}_R$$

Linear logic

- If we would like to think of proofs as programs and reduction of proofs as evaluating a program, these rules cause us a big problem.
- It turns out that their presence allows us, through the process of cut-elimination, to obtain many different reduced proofs of the same proposition.

Linear logic

- In linear logic we use two modalities, ! and ?, to mark the use of weakening and contraction on the left or right, respectively.
- We refer to ! as "of course", "bang", or "bling", and we refer to ? as "why not".

$$\frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta}\ \text{Weakening}_L \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash ?A, \Delta}\ \text{Weakening}_R$$

$$\frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ \text{Contraction}_L \qquad \frac{\Gamma \vdash ?A, ?A, \Delta}{\Gamma \vdash ?A, \Delta}\ \text{Contraction}_R$$

Linear logic

- The language of (classical) linear logic is given in Backus–Naur form as

$$A ::= p \mid p^{\perp} \mid A \otimes A \mid A \oplus A \mid A \,\&\, A \mid A \parr A \mid 1 \mid 0 \mid \top \mid \bot \mid\, !A \mid ?A.$$

- The connectives $\otimes$ and $\parr$ are called *multiplcative conjunction* and *multiplicative disjunction*, respectively.
- The connectives $\top$ and $\bot$ are also considered *multiplicative*.

Linear logic

- Note that we can now explain the interpretation of $\Gamma \vdash \Delta$ in linear logic.
- We interpret $\Gamma \vdash \Delta$ as saying that the multiplicative conjunction of Γ entails the multiplicative disjunction of Δ.

Linear logic

- The language of (classical) linear logic is given in Backus–Naur form as

$$A ::= p \mid p^{\perp} \mid A \otimes A \mid A \oplus A \mid A \mathbin{\&} A \mid A \parr A \\ \mid 1 \mid 0 \mid \top \mid \perp \mid\, !A \mid ?A.$$

- The connectives & and $\oplus$ are called *additive conjunction* and *additive disjunction*, respectively.
- The connectives 1 and 0 are also considered *additive*.

Linear logic

- Note that we have "doubled" all of the connectives from classical logic.
- To see why, consider the classical rules for conjunctions and disjunctions.

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B} \qquad \frac{\Gamma \vdash A, B}{\Gamma \vdash A \vee B}$$

$$\frac{}{\vdash 1} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash 0, \Delta}$$

Linear logic

- For the multiplicative fragment of linear logic we have corresponding rules.

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad \frac{\Gamma \vdash A, B}{\Gamma \vdash A \parr B}$$

$$\frac{}{\vdash \top} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \bot, \Delta}$$

Linear logic

- For the additive fragment of linear logic we have corresponding rules.

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \mathbin{\&} B} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B} \qquad \frac{}{\vdash 1}$$

Linear logic

- Note that while both the multiplicative and additive rules are admissible in classical logic, they behave differently here.
- In the multiplicative case the contexts Γ and Δ are both carried forward for $\otimes$ while in the additive case we need to have the same context to obtain $A \& B$.
- These are distinct in linear logic because contexts are multisets of propositions, not sets.

Linear logic

- We also have a linear notion of implication, which is defined by the formula
$$A \multimap B := A^{\perp} \parr B.$$
- As one might imagine we have the rule $\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$.
- We also have the following equivalence:
$$A \otimes B \vdash C \equiv A \vdash B \multimap C.$$

de Paiva's "Girard construction"

- We also have the following equivalence:

$$A \otimes B \vdash C \equiv A \vdash B \multimap C.$$

- This equivalence looks like the adjunction between a tensor bifunctor and an internal hom in a category:

$$A \otimes B \to C \cong A \to [B, C].$$

de Paiva's "Girard construction"

- In 1987, shortly after Jean-Yves Girard introduced linear logic, he and Valeria de Paiva met in Boulder.
- He encouraged (challenged?) her to produce a model of linear logic using category theory.
- The resulting *Girard construction* constitutes part of de Paiva's PhD thesis.

de Paiva's "Girard construction"

- We take the previous analogy forward by thinking of propositions (or contexts) as objects in a category.
- We interpret $\Gamma \vdash \Delta$ to mean that there is a morphism $\Gamma \to \Delta$.
- We would like the binary connectives to be bifunctors.

de Paiva's "Girard construction"

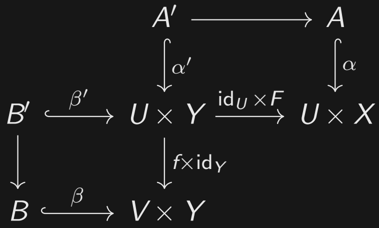
- In order to perform the Girard construction we start with a finitely complete category C.
- The category GC, our model of linear logic, has for objects relations on the objects of C.
- By definition these are (equivalence classes of) monomorphisms $\alpha\colon A \hookrightarrow U \times X$.

de Paiva's "Girard construction"

- Morphisms from $\alpha \colon A \hookrightarrow U \times X$ to $\beta \colon B \hookrightarrow V \times Y$ in GC are pairs

$$(f \colon U \to V, F \colon Y \to X)$$

such that there is a unique morphism $k \colon A' \to B'$ making a commutative triangle in the following diagram.



de Paiva's "Girard construction"

- If our α and β were set-theoretic relations this diagram tells use that there is a morphism from α to β when

$$u \; \alpha \; F(y)$$

implies that

$$f(u) \; \beta \; y.$$

- We can also describe this by saying that

$$(\mathrm{id}_U \times F)^{-1}(\alpha) \leq (f \times \mathrm{id}_Y)^{-1}(\beta).$$

de Paiva's "Girard construction"

- When C is Cartesian closed (like the category **Set**), we can define a bifunctor $\oslash$ on GC which intuitively has $(u, v)\alpha \oslash \beta(f, g)$ when $u\alpha f(v)$ and $v\beta g(u)$.
- We can also define an internal hom $[\_, \_]$ for GC such that this $\oslash$ is left adjoint to the internal hom.

de Paiva's "Girard construction"

- Assuming C is finitely complete, (even just locally) Cartesian closed, and also has stable (under pullbacks) and disjoint coproducts we can define another bifunctor: $\parr$.

de Paiva's "Girard construction"

- Theorem 3 on page 59 in de Paiva's thesis says that if we think of $\oslash$ as the multiplicative conjunction, $\parr$ as the multiplicative disjunction, $[\_, \_]$ as $\multimap$, the Cartesian product as &, and the coproduct as $\oplus$ then for each entailment $\Gamma \vdash A$ of linear logic there is a corresponding morphism $(f, F)\colon |\Gamma| \to |A|$ and vice versa.

de Paiva's "Girard construction"

- There are a couple caveats.
- The category GC actually only models linear logic with the weakened form of the rule for $\multimap$ given previously.
- Linear logic can formulated without negation, but the usual linear logic has that $A \equiv A^{\perp\perp}$.
- In GC we don't typically have that $A \cong A^{\perp\perp}$.

References

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