Universal Algebra and Boolean Semilattices

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Algebraic Structures

Definition (Cartesian power)

For a natural $r \geq 1$ we define the r^{th} *Cartesian power* of a set A to be

$$A^r := \{(a_1, \ldots, a_r) | (\forall 1 \leq i \leq r) a_i \in A\}.$$

We define $A^0 := \{()\}$ where $()$ is the empty tuple.

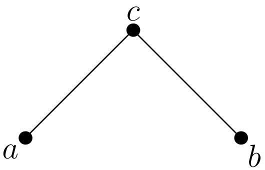
Definition (Algebra)

An *algebra* $\mathbf{A}$ is a pair $(A, F = \{f_1, \ldots, f_n\})$ where A is a nonempty set and each of the f_i is a function $f_i : A^r \rightarrow A$ for some integer $r \geq 0$. The f_i are called the *operations* of $\mathbf{A}$.

Example: An Algebra $\mathbf{A}$

Let $A = \{a, b, c\}$. Consider the algebra $\mathbf{A} = (A, \{\vee\})$ where $\vee : A^2 \to A$ is given by the following table.

$\vee$	a	b	c
a	a	c	c
b	c	b	c
c	c	c	c



Varieties

Definition (Variety)

A *variety* of algebras is a class of algebras closed under taking homomorphic images, subalgebras, and products.

For those who have had some abstract algebra, the three operators referred to in the above definition are generalizations of the same concepts for groups and rings. For finite-dimensional vector spaces they correspond to taking images under linear maps, subspaces, and direct products.

Example: The Variety of Semilattices

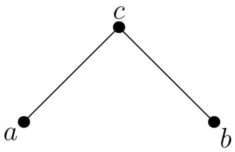
Definition (Semilattice)

A *semilattice* is an algebra of the form $\mathbf{S} = (S, \{*\})$ where for all $x, y, z \in S$ we have that $* : S^2 \to S$ satisfies

- (Associativity) $(x * y) * z = x * (y * z)$,
- (Commutativity) $x * y = y * x$, and
- (Idempotence) $x * x = x$.

Our algebra $\mathbf{A}$ from before is a semilattice.

$\vee$	a	b	c
a	a	c	c
b	c	b	c
c	c	c	c



Equational Classes

Definition (Equational class)

An *equational class* is a class consisting of all algebras which satisfy a fixed set of identities.

For example, the class of all semilattices is an equational class. One might wonder whether there are varieties which are not equational classes.

Theorem (Birkhoff's Theorem, 1935)

Every variety is an equational class.

Complex Algebras

Definition (Complex algebra)

Given an algebra $\mathbf{S} = (S, \{\cdot\})$ we define the *complex algebra* $\mathbf{S}^+$ by $\mathbf{S}^+ := (\mathrm{Sb}(S), \{\cap, \cup, \sim, \odot, \varnothing, S\})$ where $\mathrm{Sb}(S)$ is the power set of S, $\cap$ and $\cup$ are set intersection and union, respectively, and $\sim$ is set difference. Given $X, Y \subset S$ we define
$X \odot Y := \{x \cdot y | x \in X \text{ and } y \in Y\}$.

Given a variety of algebras we would like to understand which identities are satisfied by its complex algebras.

Example: The Complex Algebra $\mathbf{A}^+$

Below is the operation table for $\odot$ in $\mathbf{A}^+$.

$\odot$	$\varnothing$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	A
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
$\{a\}$	$\varnothing$	$\{a\}$	$\{c\}$	$\{c\}$	$\{a,c\}$	$\{a,c\}$	$\{c\}$	$\{a,c\}$
$\{b\}$	$\varnothing$	$\{c\}$	$\{b\}$	$\{c\}$	$\{b,c\}$	$\{c\}$	$\{b,c\}$	$\{b,c\}$
$\{c\}$	$\varnothing$	$\{c\}$	$\{c\}$	$\{c\}$	$\{c\}$	$\{c\}$	$\{c\}$	$\{c\}$
$\{a,b\}$	$\varnothing$	$\{a,c\}$	$\{b,c\}$	$\{c\}$	A	$\{a,c\}$	$\{b,c\}$	A
$\{a,c\}$	$\varnothing$	$\{a,c\}$	$\{c\}$	$\{c\}$	$\{a,c\}$	$\{a,c\}$	$\{c\}$	$\{a,c\}$
$\{b,c\}$	$\varnothing$	$\{c\}$	$\{b,c\}$	$\{c\}$	$\{b,c\}$	$\{c\}$	$\{b,c\}$	$\{b,c\}$
A	$\varnothing$	$\{a,c\}$	$\{b,c\}$	$\{c\}$	A	$\{a,c\}$	$\{b,c\}$	A

Boolean Algebras

Definition (Boolean algebra)

A *Boolean algebra* $\mathbf{B} = (B, \{\wedge, \vee, ', 0, 1\})$ is an algebra such that for all $x, y, z \in B$ we have

- (Associativity) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$,
- (Commutativity) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$,
- (Absorption) $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$,
- (Identity) $x \wedge 1 = x$ and $x \vee 0 = x$,
- (Distributivity) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, and
- (Complements) $x \wedge x' = 0$ and $x \vee x' = 1$.

Boolean Semilattices

Definition (Boolean semilattice)

A *Boolean semilattice* $\mathbf{B} = (B, \{\wedge, \vee, ', \cdot, 0, 1\})$ is a Boolean algebra such that for all $x, y, z \in B$ we have

- (Normality) $x \cdot 0 = 0 = 0 \cdot x$,
- (Additivity) $x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$ and $(y \vee z) \cdot x = (y \cdot x) \vee (z \cdot x)$,
- (Associativity) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
- (Commutativity) $x \cdot y = y \cdot x$, and
- (Square-increasing law) $x \vee (x \cdot x) = x \cdot x$.

Action Matrices

Consider the Boolean semilattice we call $\mathbf{B}_2$. The underlying set of this Boolean semilattice is $B = \{a, b, 0 = a \wedge b, 1 = a \vee b\}$.

$\cdot$	a	b
a	a	b
b	b	$a \vee b$

We can define a function $f_b : B \to B$ by $f(x) := b \cdot x$. Since $\cdot$ distributes over $\vee$ (additivity), we find that f_b can be viewed as a linear transformation.

$$f_b(b) = b \cdot b = a \vee b \quad \longrightarrow \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Action Matrices

We can linearize the operations on a Boolean semilattice to find identities. Let $M_a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and let $M_b = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Left multiplication by M_a corresponds to left multiplication by a and similarly M_b corresponds to left multiplication by b. We call the matrix M_x associated to x in this way an *action matrix* for x.

$$\begin{array}{c|cc} \cdot & a & b \\ \hline a & a & b \\ b & b & a \vee b \end{array} \longrightarrow \begin{array}{c|cc} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{array}$$

Identity Computation

Observe that the minimum polynomial for M_a is $t - 1$, while the minimum polynomial for M_b is $t^2 - t - 1$. The least common multiple of these polynomials is $(t-1)(t^2 - t - 1)$, so for $M \in \{M_a, M_b\}$ we have that M satisfies

$$M^3 + I = 2M^2.$$

This implies that for any x and y from $\mathbf{B}_2$ we have

$$(x \cdot x \cdot x \cdot y) \vee y = x \cdot x \cdot y.$$

Modal Logic S4.3 and the Variety IBSL

The variety of *idempotent Boolean semilattices* (IBSL) consists of those Boolean semilattices which for all x satisfy $x = x \cdot x$. We were interested in which smaller varieties were contained in IBSL. As it turns out, IBSL is term-equivalent to a modal logic called S4.3, which has already been studied. Previous work by Kit Fine in the 1970s tells us that every variety contained in IBSL can be defined by a finite set of identities.

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