

A multi-linear geometric estimate

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Introduction

- Erdős and Szemerédi made the following significant conjecture in additive number theory: If A is a finite set of integers with $|A| = n$ then either $A + A$ or $A \cdot A$ must have size at least $C_\epsilon n^{2-\epsilon}$ for any $\epsilon > 0$.
- The sum-product conjecture naturally leads one to the consideration of the size of sets of the form $A \cdot A + A \cdot A$.

Introduction

- Hart and Iosevich previously showed that if $E \subset \mathbb{F}_q^d$ with $|E| > q^{\frac{d+1}{2}}$ then $\mathbb{F}_q^* \subset \varpi(E^2)$ where ϖ is any non-degenerate bilinear form.
- This estimate can be used to show that if $A \subset \mathbb{F}_q^*$ is sufficiently large then

$$\mathbb{F}_q^* \subset dA^2 = A \cdot A + \cdots + A \cdot A.$$

- We generalize this estimate to the case that ϖ is a multi-linear form.

Talk outline

- Preliminaries on forms
- The main result
- Some caveats
- Applications
- Proof sketch

Preliminaries on forms

- Given two vector spaces V and W let $\boxtimes: V \times W \rightarrow V \otimes W$ denote the canonical map taking (v, w) to $v \otimes w$.
- Note that when E is a subspace of V both $E^{\boxtimes n}$ and $E^{\otimes n}$ are defined and are in general distinct.
- An n -linear form is a linear transformation $\varpi: V^{\otimes n} \rightarrow \mathbb{F}$ where V is an \mathbb{F} -vector space.
- For example, the usual dot product over \mathbb{F}_q^d is a bilinear ($n = 2$) form.

Preliminaries on forms

Definition (Space of multi-linear forms)

Given a vector space V over a field \mathbb{F} and some $n \in \mathbb{N}$ we denote by $\text{Form}(V, n)$ the dual \mathbb{F} -vector space to $V^{\otimes n}$. That is, $\text{Form}(V, n) := \text{Hom}(V^{\otimes n}, \mathbb{F})$.

Preliminaries on forms

Definition (Level set)

Given an \mathbb{F} -vector space V , a form $\varpi \in \text{Form}(V, n)$, $E \subset V$, $t \in \mathbb{F}$ we define the t -level set of ϖ (with respect to E) to be

$$L_t := \left\{ (z, w) \in E^{\boxtimes(n-1)} \times E \mid \varpi(z, w) = t \right\}$$

and we define $\nu(t) := |L_t|$.

Preliminaries on forms

Definition (Evaluation map)

Given a vector space V , some $n \in \mathbb{N}$, some $k \in [n]$, and subspaces $A \leq V^{\otimes(n-1)}$ and $B \leq V$ the k^{th} evaluation map on (A, B) is

$$\text{eval}_{k,A,B}: \text{Form}(V, n) \otimes B \rightarrow \text{Hom}(A, \mathbb{F})$$

is given by

$$(\text{eval}_{k,A,B}(\varpi \otimes y))(x_1, \dots, x_{n-1}) := \varpi(x_1, \dots, x_{k-1}, y, x_{k+1}, \dots, x_{n-1}).$$

Preliminaries on forms

Definition ((A, B)-non-degenerate form)

Given a form $\varpi \in \text{Form}(V, n)$ and subspaces $A \leq V^{\otimes(n-1)}$ and $B \leq V$ we say that ϖ is *(A, B)-non-degenerate* in the k^{th} coordinate when $\text{Ker}(\text{eval}_{k,A,B}^{\varpi}) = 0$.

Preliminaries on forms

- Let $V = \mathbb{F}_q^d$, $A = V^{\otimes(n-1)}$, $B = V$, $n = 3$, and

$$\varpi(x, y, z) = x_1y_1z_1 + x_2y_2z_2 + \cdots + x_dy_dz_d.$$

It is not difficult to see that this form is (A, B) -non-degenerate.

- If we keep B the same, change A to $W^{\otimes(n-1)}$, where

$$W = \left\{ x \in \mathbb{F}_q^d \mid x_1 = 0 \right\},$$

and use the same form as above, we get an (A, B) -degenerate form.

Preliminaries on forms

Definition (Projective index)

Given $E \subset \mathbb{F}_q^d$ we say that E has *projective index* α when

$$\frac{|\{(a, w) \in (\mathbb{F}_q^* \setminus \{1\}) \times \mathbb{F}_q^d \mid w, aw \in E\}|}{(q-2)|E|} \geq \alpha.$$

The main result

Theorem (A., Iosevich (2021))

Suppose that $\varpi \in \text{Form}(q, d, n)$ for some $n \geq 2$, that $E \subset \mathbb{F}_q^d$, and that E has projective index α . If there exists an r -dimensional subspace A of $(\mathbb{F}_q^d)^{\otimes(n-1)}$ and a subspace B of \mathbb{F}_q^d such that

- 1 $E^{\boxtimes(n-1)} \subset A$,
- 2 $E \subset B$
- 3 ϖ is (A, B) -non-degenerate, and

- 4 $|E| > q^{\frac{r+n-1}{n}} \left(1 - \alpha \left(1 - \frac{2}{q}\right)\right)^{\frac{1}{n}}$

then $\mathbb{F}_q^* \subset \varpi(E^n)$.

Some caveats

- This result extends the 2008 Hart-Iosevich result by incorporating the projective index α .
- This is only a constant improvement, and we're not in the business of constants.

Some caveats

- This result extends the 2008 Hart-Iosevich result for $d = 2$ by incorporating the projective index α .
- If I recall correctly, trying to exploit this to get a better exponent for $d = 2$ is a losing battle.

Some caveats

- When $\dim(\text{Span}(E)) = \ell$ we need

$$\ell^{n-1} - n\ell + n < 2 - \log_q(q - \alpha(q - 2))$$

in order for our exponent to be nontrivial.

- There are only two cases where this occurs: $n = 2$ and d is anything, or $n = 3$ and $d = 2$.

Applications

- When I spoke about this for Virginia Tech, I gave the following kind of example applications when $n = 3$ and $d = 2$.

Applications

- Let $q = 160001$.
- Every nonzero member of \mathbb{F}_q^* may be written as

$$\varpi(h_1\gamma_1(1, h_2\gamma_2), h_3\gamma_3(1, h_4\gamma_4), h_5\gamma_5(1, h_6\gamma_6))$$

where the h_i are from a fixed set H consisting of 16 coset representatives of the subgroup Γ of \mathbb{F}_q^* of order $\frac{q-1}{20} = 8000$ and the γ_i are members of Γ .

Applications

- Each member of \mathbb{F}_q^* is of the form

$$h_1 h_3 h_5 \psi_1 (1 + h_2 h_4 h_6 \psi_2)$$

where ψ_1 and ψ_2 are 20th powers in \mathbb{F}_q and the h_i belong to H .

- Moreover,

$$A \cdot A \cdot A + A \cdot A \cdot A \cdot A \cdot A \cdot A \supset \mathbb{F}_q^*$$

when $A = H\Gamma$.

Applications

Theorem (Cauchy-Davenport)

Given $A, B \subset \mathbb{F}_p$ we have that

$$|A + B| \geq \min(|A| + |B| - 1, p).$$

Applications

- In our example we had

$$A \cdot A \cdot A + A \cdot A \cdot A \cdot A \cdot A \cdot A \supset \mathbb{F}_q^*$$

when $A = H\Gamma$.

- Note that $|A| = 16(8000) = 128000$,

$$|A \cdot A \cdot A| \geq |A|,$$

and

$$|A \cdot A \cdot A \cdot A \cdot A \cdot A| \geq |A|.$$

Applications

- Cauchy-Davenport yields

$$\begin{aligned} |A \cdot A \cdot A + A \cdot A \cdot A \cdot A \cdot A \cdot A| &\geq \min(|A| + |A| - 1, 160001) \\ &= \min(255999, 160001) \\ &= 160001. \end{aligned}$$

- Cauchy-Davenport is stronger here.

Proof sketch

Theorem (A., Iosevich (2021))

Suppose that $\varpi \in \text{Form}(q, d, n)$ for some $n \geq 2$, that $E \subset \mathbb{F}_q^d$, and that E has projective index α . If there exists an r -dimensional subspace A of $(\mathbb{F}_q^d)^{\otimes(n-1)}$ and a subspace B of \mathbb{F}_q^d such that

- 1 $E^{\boxtimes(n-1)} \subset A$,
- 2 $E \subset B$
- 3 ϖ is (A, B) -non-degenerate, and

- 4 $|E| > q^{\frac{r+n-1}{n}} \left(1 - \alpha \left(1 - \frac{2}{q}\right)\right)^{\frac{1}{n}}$

then $\mathbb{F}_q^* \subset \varpi(E^n)$.

Proof sketch

- Write

$$\nu(t) = \sum_{\substack{z \in E^{\boxtimes(n-1)} \\ w \in E}} q^{-1} \sum_{s \in \mathbb{F}_q} \chi(s(\varpi(z, w) - t)).$$

- Thus,

$$\nu(t) = q^{-1} |E^{\boxtimes(n-1)}| |E| + R$$

where

$$R := \sum_{\substack{z \in E^{\boxtimes(n-1)} \\ w \in E}} q^{-1} \sum_{s \in \mathbb{F}_q^*} \chi(s(\varpi(z, w) - t)).$$

Proof sketch

- View R as a sum in z and apply Cauchy-Schwarz.
- We find that $R^2 \leq U + V$ where

$$U = \left| E^{\boxtimes(n-1)} \right| q^{r-2} \sum_{\substack{s, s' \in \mathbb{F}_q^* \\ w, w' \in \mathbb{F}_q^d \\ sw = s' w'}} \chi(t(s' - s)) E(w) E(w')$$

and

$$V = \left| E^{\boxtimes(n-1)} \right| q^{-2} \sum_{\substack{s, s' \in \mathbb{F}_q^* \\ w, w' \in E \\ sw \neq s' w'}} \chi(t(s' - s)) \sum_{z \in A} \chi(\varpi(z, sw - s' w'))$$

- The (A, B) -nondegeneracy of ϖ and orthogonality of χ gives $V = 0$.

Proof sketch

- We have $R^2 \leq U = C + D$ where

$$C := \left| E^{\boxtimes(n-1)} \right| q^{r-2} \sum_{\substack{s, s' \in \mathbb{F}_q^* \\ w, w' \in \mathbb{F}_q^d \\ sw = s'w' \\ s \neq s'}} \chi(t(s' - s)) E(w) E(w')$$

and

$$D := \left| E^{\boxtimes(n-1)} \right| q^{r-2} \sum_{\substack{s, s' \in \mathbb{F}_q^* \\ w, w' \in \mathbb{F}_q^d \\ sw = s'w' \\ s = s'}} \chi(t(s' - s)) E(w) E(w').$$

- Without using the projective index α we can just note that $C < 0$, but in general this is not enough.

Proof sketch

- Since

$$C \leq - \left| E^{\boxtimes(n-1)} \right| |E| q^{r-1} \alpha \left(1 - \frac{2}{q} \right)$$

and

$$D = \left| E^{\boxtimes(n-1)} \right| |E| q^{r-1}$$

we have $\nu(t) > 0$ and the result follows.

References

- **Derrick Hart and Alex Iosevich.** “Sums and products in finite fields: an integral geometric viewpoint.” In: *Radon transforms, geometry, and wavelets*. Vol. 464. *Contemp. Math.* Providence, RI: Amer. Math. Soc., 2008, pp. 129–135