Monoid representations and partitions

Charlotte Aten

University of Denver

2023 October 20

Introduction

- Today I'll discuss how I came across the formula

$$p(n) = \frac{1}{n!} \sum_{g \in V_n} \left(\prod_{k=1}^{n} (k-1)!^{g(k)} (g(k))! \binom{n - \sum_{s=1}^{k-1} sg(s)}{g(k)} \prod_{v=1}^{g(k)} \binom{n - \sum_{s=1}^{k-1} sg(s) - g(k) - (v-1)(k-1)}{k-1} \right)$$

when I was a graduate student.

Introduction

- I think the formula may be interesting, as it only involves the partition number $p(n)$ and elementary operations.
- However, there is a summation over a set of functions V_n, and this set is closely related to the partitions of n.

Talk outline

- Monoid representations
- Idempotents
- Burnside's Lemma
- Speculation

Monoid representations

- A *representation* of a monoid A on a set X is a homomorphism $\rho\colon A \to T(X)$ where $T(X)$ is the full transformation monoid of maps from X to itself.

Monoid representations

- There is a natural action of the permutation group $\Sigma(X)$ on the set of such representations $R_A(X)$.
- This action is given by the conjugation

$$\rho^{\sigma}(a) := \sigma\rho(a)\sigma^{-1}$$

where $\sigma \in \Sigma(X)$ and $a \in A$.

Idempotents

- Consider the free idempotent monoid on one generator

$$B := M(\{x\})/\langle(x, x^2)\rangle.$$

- Let e and b denote the equivalence classes of the identity and x under $\langle(x, x^2)\rangle$, respectively.

Idempotents

- A representation of B on a set X is basically a choice of an idempotent map $f\colon X \to X$. That is, f satisfies $f \circ f = f$.
- We then find that $\Sigma(X)$ acts on the set $I(X)$ of idempotent self-maps of X by conjugation.

Burnside's Lemma

- We can find the stabilizer subgroup for any $f \in I(X)$.
- Define $\eta\colon \mathrm{Im}(f) \to \mathrm{Sb}(X)$ by

$$\eta(x) := \left\{\, y \in \mathrm{Im}(f) \;\middle|\; \left|f^{-1}(x)\right| = \left|f^{-1}(y)\right| \,\right\}.$$

Burnside's Lemma

- Given $U \in \mathrm{Im}(\eta)$ choose a representative $x_U \in U$. Let $H_U := \Sigma(U)$. Define $A_U := \Sigma(f^{-1}(x_U) \setminus \{x_U\})$ and define $K_U := A_U^U$. Let $\alpha \colon H_U \to \mathrm{Aut}(K_U)$ be given by

$$\alpha(h)((a_u)_{u \in U}) := (a_{h(u)})_{u \in U}.$$

Burnside's Lemma

- Let G_U be the group whose multiplication is given by

$$((a_u)_{u\in U}, \varphi_1)((b_u)_{u\in U}, \varphi_2) := (\alpha(\varphi_2)((a_u)_{u\in U})(b_u)_{u\in U}, \varphi_1\varphi_2).$$

- This is a wreath product of two symmetric groups.

Burnside's Lemma

- The stabilizer of $f \in I(X)$ satisfies

$$\mathbf{Stab}(f) \cong \prod_{U \in \mathrm{Im}(\eta)} G_U.$$

Burnside's Lemma

- Burnside's Lemma then gives us that

$$p(|X|) = \frac{1}{|X|\,!} \sum_{f \in I(X)} |\mathrm{Stab}(f)|\,.$$

Burnside's Lemma

- We compute

$$|\mathrm{Stab}(f)| = \prod_{k=1}^{n} (k-1)!^{g(k)}\,(g(k))!\,.$$

where $g(k)$ is the number of parts with size k in the partition induced by f.

Burnside's Lemma

- The number of parts with the same g is

$$\prod_{k=1}^{n} \binom{n - \sum_{s=1}^{k-1} sg(s)}{g(k)}.$$

$$\prod_{v=1}^{g(k)} \binom{n - \sum_{s=1}^{k-1} sg(s) - g(k) - (v-1)(k-1)}{k-1}$$

Burnside's Lemma

- Taking V_n to be

$$\left\{ g\colon [n] \to \{0, \ldots, n\} \,\middle|\, \sum_{k=1}^{n} k g(k) = n \right\}$$

we obtain the desired formula:

$$p(n) = \frac{1}{n!} \sum_{g \in V_n} \left(\prod_{k=1}^{n} (k-1)!^{g(k)} (g(k))! \binom{n - \sum_{s=1}^{k-1} s g(s)}{g(k)} \prod_{v=1}^{g(k)} \binom{n - \sum_{s=1}^{k-1} s g(s) - g(k) - (v-1)(k-1)}{k-1} \right)$$