Distributive lattices in rock-paper-scissors

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Introduction

- I will show by example how to construct games for several players which have many of the familiar properties of rock-paper-scissors.
- These constructions will all begin with the cyclic groups $\mathbb{Z}/m\mathbb{Z}$ and will yield algebras $\mathbf{A} := (A, f)$ with a single n-ary operation.
- The congruence lattices of these algebras are all distributive.

Introduction

We will view the game of RPS as a magma $\mathbf{A} := (A, f)$. We let $A := \{r, p, s\}$ and define a binary operation $f \colon A^2 \to A$ where $f(x, y)$ is the winning item among $\{x, y\}$.

	r	p	s
r	r	p	r
p	p	p	s
s	r	s	s

A game from modular arithmetic

First we list the orbits of $\binom{\mathbb{Z}_3}{1}$ and $\binom{\mathbb{Z}_3}{2}$ under this action of $\mathbb{Z}_3$.

0	01
1	12
2	20

A game from modular arithmetic

Next, we choose a representative for each orbit, say the first one in each row of this table.

0	01
1	12
2	20

A game from modular arithmetic

Choose from each representative a particular element. For example, if our representative is 02 we may choose 0 as our special element. We also could have chosen 2, but not 1.

$0 \mapsto 0$	$01 \mapsto 1$
1	12
2	20

A game for three players

Use the left-addition action of $\mathbb{Z}_3$ to extend these choices to all members of $\binom{\mathbb{Z}_3}{1}$ and $\binom{\mathbb{Z}_3}{2}$.

$$\begin{array}{c|c} 0 \mapsto 0 & 01 \mapsto 1 \\ 1 \mapsto 1 & 12 \mapsto 2 \\ 2 \mapsto 2 & 20 \mapsto 0 \end{array}$$

A game from modular arithmetic

We can read off a definition for the operation $f\colon \mathbb{Z}_3^2 \to \mathbb{Z}_3$ from this table. For example, we take $12 \mapsto 2$ to indicate that

$$f(1,2) = f(2,1) = 2.$$

$$\begin{array}{c|c} 0 \mapsto 0 & 01 \mapsto 1 \\ 1 \mapsto 1 & 12 \mapsto 2 \\ 2 \mapsto 2 & 20 \mapsto 0 \end{array}$$

A game from modular arithmetic

The original game of RPS is isomorphic to the resulting magma $(\mathbb{Z}_3, f)$.

	0	1	2
0	0	1	0
1	1	1	2
2	0	2	2

A game for three players

- We now show how to construct a game for three players.
- This will be a ternary magma $(A, f\colon A^3 \to A)$.
- Since $n = 3$ in this case we must have that $|A| \geq 5$.
- Our construction will use the left-addition action of $\mathbb{Z}_5$ on itself.
- We will produce an operation $f\colon \mathbb{Z}_5^3 \to \mathbb{Z}_5$ with $w + f(x, y, z) = f(w + x, w + y, w + z)$ for any $w \in \mathbb{Z}_5$.
- Thus, we need only define f on a representative of each orbit of $\binom{\mathbb{Z}_5}{1}$, $\binom{\mathbb{Z}_5}{2}$, and $\binom{\mathbb{Z}_5}{3}$ under this action of $\mathbb{Z}_5$.

A game for three players

First we list the orbits of $\binom{\mathbb{Z}_5}{1}$, $\binom{\mathbb{Z}_5}{2}$, and $\binom{\mathbb{Z}_5}{3}$ under this action of $\mathbb{Z}_5$.

0	01	02	012	013
1	12	13	123	124
2	23	24	234	230
3	34	30	340	341
4	40	41	401	402

A game for three players

Next, we choose a representative for each orbit, say the first one in each row of this table.

0	01	02	012	013
1	12	13	123	124
2	23	24	234	230
3	34	30	340	341
4	40	41	401	402

A game for three players

Choose from each representative a particular element. For example, if our representative is 013 we may choose 0 as our special element. We also could have chosen 1 or 3, but not 2 or 4.

$0 \mapsto 0$	$01 \mapsto 1$	$02 \mapsto 0$	$012 \mapsto 0$	$013 \mapsto 0$
1	12	13	123	124
2	23	24	234	230
3	34	30	340	341
4	40	41	401	402

A game for three players

Use the left-addition action of $\mathbb{Z}_5$ to extend these choices to all members of $\binom{\mathbb{Z}_5}{1}$, $\binom{\mathbb{Z}_5}{2}$, and $\binom{\mathbb{Z}_5}{3}$.

$0 \mapsto 0$	$01 \mapsto 1$	$02 \mapsto 0$	$012 \mapsto 0$	$013 \mapsto 0$
$1 \mapsto 1$	$12 \mapsto 2$	$13 \mapsto 1$	$123 \mapsto 1$	$124 \mapsto 1$
$2 \mapsto 2$	$23 \mapsto 3$	$24 \mapsto 2$	$234 \mapsto 2$	$230 \mapsto 2$
$3 \mapsto 3$	$34 \mapsto 4$	$30 \mapsto 3$	$340 \mapsto 3$	$341 \mapsto 3$
$4 \mapsto 4$	$40 \mapsto 0$	$41 \mapsto 4$	$401 \mapsto 4$	$402 \mapsto 4$

A game for three players

We can read off a definition for the operation $f\colon \mathbb{Z}_5^3 \to \mathbb{Z}_5$ from this table. For example, we take $24 \mapsto 2$ to indicate that

$$f(2,4,4) = f(4,2,4) = f(4,4,2) = f(4,2,2) = f(2,4,2) = f(2,2,4) = 2.$$

$0 \mapsto 0$	$01 \mapsto 1$	$02 \mapsto 0$	$012 \mapsto 0$	$013 \mapsto 0$
$1 \mapsto 1$	$12 \mapsto 2$	$13 \mapsto 1$	$123 \mapsto 1$	$124 \mapsto 1$
$2 \mapsto 2$	$23 \mapsto 3$	$24 \mapsto 2$	$234 \mapsto 2$	$230 \mapsto 2$
$3 \mapsto 3$	$34 \mapsto 4$	$30 \mapsto 3$	$340 \mapsto 3$	$341 \mapsto 3$
$4 \mapsto 4$	$40 \mapsto 0$	$41 \mapsto 4$	$401 \mapsto 4$	$402 \mapsto 4$

A game for three players

The Cayley table for the 3-magma $\mathbf{A} := (\mathbb{Z}_5, f)$ obtained from this choice of f is given below.

0	0	1	2	3	4
0	0	1	0	3	0
1	1	1	0	0	4
2	0	0	0	2	4
3	3	0	2	3	3
4	0	4	4	3	0

1	0	1	2	3	4
0	1	1	0	0	4
1	1	1	2	1	4
2	0	2	2	1	1
3	0	1	1	1	3
4	4	4	1	3	4

2	0	1	2	3	4
0	0	0	0	2	4
1	0	2	2	1	1
2	0	2	2	3	2
3	2	1	3	3	2
4	4	1	2	2	2

3	0	1	2	3	4
0	3	0	2	3	3
1	0	1	1	1	3
2	2	1	3	3	2
3	3	1	3	3	4
4	3	3	2	4	4

4	0	1	2	3	4
0	0	4	4	3	0
1	4	4	1	3	4
2	4	1	2	2	2
3	3	3	2	4	4
4	0	4	2	4	4

Regular RPS magmas

- A regular RPS(m, n) magma is a magma so constructed by considering the action of $\mathbb{Z}_m$ on the 1-, 2-, ..., and n-sets in $\mathbb{Z}_m$.
- Denoting our choices of orbit representatives and special elements by λ, such a magma is denoted by $(\mathbb{Z}_m)_n(\lambda)$.

Congruences

Theorem

Let $\theta \in \mathrm{Con}(\mathbf{A})$ for a regular $\mathrm{RPS}(m, n)$ magma $\mathbf{A} := \mathbf{G}_n(\lambda)$. Given any $a \in A$ we have that $a/\theta = aH$ for some subgroup $\mathbf{H} \leq \mathbf{G}$.

- One can show by using 2-divisibility that the principal congruence $\theta := \mathrm{Cg}(\{(e, a)\})$ has only one nontrivial class, which is e/θ. This class contains $\mathrm{Sg}^{\mathbf{G}}(\{a\})$.

Congruences

Theorem

Let $\theta \in \operatorname{Con}(\mathbf{A})$ for a regular $\operatorname{RPS}(m, n)$ magma $\mathbf{A} := \mathbf{G}_n(\lambda)$. Given any $a \in A$ we have that $a/\theta = aH$ for some subgroup $\mathbf{H} \leq \mathbf{G}$.

- Any congruence $\theta \in \operatorname{Con}(\mathbf{A})$ has for e/θ a union of cyclic subgroups of $\mathbf{G}$. Suppose that $a, b \in e/\theta$ and $ab \notin e/\theta$.
- Note that $\theta \geq \operatorname{Cg}(\{(e, a), (e, b^{-1})\})$. Observe that

$$\begin{aligned}
\operatorname{Cg}(\{(e, a), (e, b^{-1})\}) &= b^{-1} \operatorname{Cg}(\{(b, ba), (b, e)\}) \\
&\geq b^{-1} \operatorname{Cg}(\{(e, ba)\}) \\
&\geq b^{-1} \operatorname{Cg}(\{(e, baba)\}) \\
&\geq \operatorname{Cg}(\{(b^{-1}, aba)\})
\end{aligned}$$

so we have that e/θ contains aba.

Congruences

Theorem

Let $\theta \in \mathrm{Con}(\mathbf{A})$ for a regular $\mathrm{RPS}(m, n)$ magma $\mathbf{A} := \mathbf{G}_n(\lambda)$. Given any $a \in A$ we have that $a/\theta = aH$ for some subgroup $\mathbf{H} \leq \mathbf{G}$.

- We have $\langle a \rangle, \langle b \rangle \subset e/\theta$ and $ab \notin e/\theta$ yet $aba \in e/\theta$.
- Since θ is a congruence either ab dominates everything in e/θ ($f(ab, x) = ab$ for all $x \in e/\theta$, which we write as $ab \to x$) or everything in e/θ dominates ab.
- In the former case, we have $ab \to aba$ so $e \to a$.
- We also have $ab \to e$ so $e \to b^{-1}a^{-1}$.
- This implies that $b^{-1} \to b^{-1}a^{-1}$ and hence $e \to a^{-1}$, which is impossible since $e \to a$.
- The argument in the latter case is identical.
- Thus, e/θ is a subgroup of $\mathbf{G}$.

λ-convex subgroups

Definition (λ-convex subgroup)

Given a group $\mathbf{G}$, an n-sign function $\lambda \in \mathrm{sgn}_n(\mathbf{G})$, and a subgroup $\mathbf{H} \leq \mathbf{G}$ we say that $\mathbf{H}$ is λ-*convex* when there exists some $a \in G$ such that $a/\theta = aH$ for some $\theta \in \mathrm{Con}(\mathbf{G}_n(\lambda))$.

λ-convex subgroups

Proposition

Let $\mathbf{G}$ *be a finite group of order* m *and let* $n < \varpi(m)$. *Take* $\lambda \in \mathrm{sgn}_n(\mathbf{G})$ *and* $\mathbf{H} \leq \mathbf{G}$. *The following are equivalent:*

1. *The subgroup* $\mathbf{H}$ *is* λ*-convex.*
2. *There exists a congruence* $\psi \in \mathrm{Con}(\mathbf{G}_n(\lambda))$ *such that* $e/\psi = H$.
3. *Given* $1 \leq k \leq n-1$ *and* $b_1, \ldots, b_k \notin H$ *either* $e \to \{b_1 h_1, \ldots, b_k h_k\}$ *for every choice of* $h_1, \ldots, h_k \in H$ *or* $\{b_1 h_1, \ldots, b_k h_k\} \to e$ *for every choice of* $h_1, \ldots, h_k \in H$.

λ-convex subgroups

Theorem

Suppose that $\mathbf{H}, \mathbf{K} \leq \mathbf{G}$ *are both* λ*-convex. We have that* $\mathbf{H} \leq \mathbf{K}$ *or* $\mathbf{K} \leq \mathbf{H}$*.*

λ-coset poset

Definition (λ-coset poset)

Given $\lambda \in \mathrm{sgn}_n(\mathbf{G})$ set

$$P_\lambda := \{\, aH \mid a \in G \text{ and } \mathbf{H} \text{ is } \lambda\text{-convex} \,\}$$

and define the *λ-coset poset* to be $\mathbf{P}_\lambda := (P_\lambda, \subset)$.

Lattices of maximal antichains

- Dilworth showed that the maximal antichains of a finite poset form a distributive lattice.
- Freese (1974) gives a succinct treatment of this.
- Given a finite poset $\mathbf{P} := (P, \leq)$ let $\mathbf{L}(\mathbf{P})$ be the lattice whose elements are maximal antichains in $\mathbf{P}$ where if $U, V \in L(\mathbf{P})$ then we say that $U \leq V$ in $\mathbf{L}(\mathbf{P})$ when for every $u \in U$ there exists some $v \in V$ such that $u \leq v$ in $\mathbf{P}$.

Theorem

We have that $\mathbf{Con}(\mathbf{G}_n(\lambda)) \cong \mathbf{L}(\mathbf{P}_\lambda)$.

Thank you.