My Hawaiian Earring

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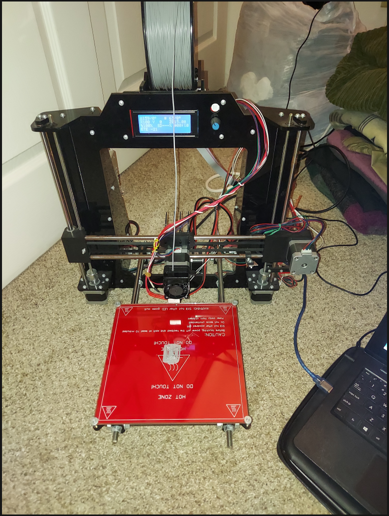
Introduction

- In the summer of 2018, at the end of my first year of graduate school, I was invited to a conference on universal algebra and lattice theory in Hawaii.
- The conference was called «Algebras and Lattices in Hawaii» and I was to give a talk on a multiplayer version of rock-paper-scissors.
- I designed an earring for the conference which depicts an object in lattice theory called the free distributive lattice on three generators.

Introduction

- The previous summer I had bought a DIY 3D printer kit based on the RepRap Prusa i3.
- I had a friend who had recently taken a class on microcontrollers at Monroe Community College and we built the printer together.
- As long as I could produce an appropriate 3D object file I could use the printer to make my earring.

The printer in 2021



Talk outline

- Posets and lattices
- Distributive lattices
- Free distributive lattices
- Drawing the earring
- Printing the earring

Posets and lattices

Definition (Poset)

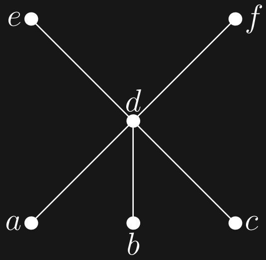
A *poset* $\mathbf{P} := (P, \leq)$ consists of a set P along with *partial order* $\leq$ on P which must be

1. reflexive (for each $x \in P$ we have $x \leq x$),
2. antisymmetric (if $x \leq y$ and $y \leq x$ then $x = y$), and
3. transitive (if $x \leq y$ and $y \leq z$ then $x \leq z$).

- For example, $(\mathbb{N}, \leq)$ is a poset with the usual definition of $\leq$ for natural numbers.

Posets and lattices

- In order to depict a poset we may use a *Hasse diagram*, which is a graph whose vertices correspond to the elements of the poset and whose edges indicate the ordering.
- The Hasse diagram of the poset $\mathbf{P} := (P, \leq)$ on $P := \{a, b, c, d, e, f\}$ with $a < d < e$, $b < d < f$, and $c < d$ is depicted below.

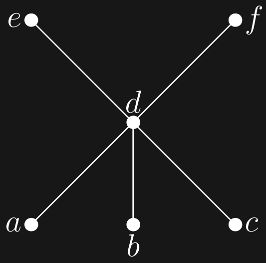


Posets and lattices

- Given a poset $\mathbf{P} := (P, \leq)$ we denote by $a \ll X$ the statement «for all $x \in X$ we have that $a \leq x$».
- In the case that $a \ll X$ we say that a is a *lower bound* for X.
- We say that a lower bound a of X is the *greatest lower bound* (or *infimum*) of X when for all $p \in P$ we have that if $p \ll X$ then $p \leq a$.
- We can similarly define *upper bound* and *least upper bound* (or *supremum*).

Posets and lattices

- In the poset **P** depicted previously we have that $a \ll \{e, f\}$.
- We also have that $b \ll \{e, f\}$, $c \ll \{e, f\}$, and $d \ll \{e, f\}$.
- Since d is the greatest among these lower bounds we have that $d = \inf(\{e, f\})$.
- Similarly, $d = \sup(\{a, b\})$.
- Since e and f have no common upper bound $\sup(\{e, f\})$ does not exist.



Posets and lattices

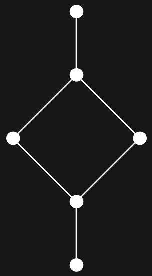
Definition (Lattice)

A *lattice* is a poset $\mathbf{P} := (P, \leq)$ in which every pair of elements of P has a supremum and infimum.

- For example, $(\mathbb{N}, \leq)$ is a lattice with the usual definition of $\leq$ for natural numbers.
- In this lattice $\inf(\{x, y\}) = \min(\{x, y\})$ and $\sup(\{x, y\}) = \max(\{x, y\})$.

Posets and lattices

- The poset **P** depicted previously is not a lattice since $\sup(\{e, f\})$ does not exist.
- The poset whose Hasse diagram is pictured below is a lattice.



Posets and lattices

Definition (Lattice)

A *lattice* $\mathbf{L} := (L, \wedge, \vee)$ consists of a set L and binary operations $\wedge$ and $\vee$ such that for any $x, y, z \in L$ we have

1. (idempotence) $x \wedge x = x$ and $x \vee x = x$,
2. (commutativity) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$,
3. (associativity)

$$(x \wedge y) \wedge z = x \wedge (y \wedge z) \text{ and } (x \vee y) \vee z = x \vee (y \vee z),$$

   and
4. (absorption)

$$x \wedge (x \vee y) = x \text{ and } x \vee (x \wedge y) = x.$$

Posets and lattices

- To make a lattice $\mathbf{L} := (L, \wedge, \vee)$ from a lattice $\mathbf{P} := (P, \leq)$ define $x \wedge y := \inf(\{x, y\})$ and $x \vee y := \sup(\{x, y\})$.
- To make a lattice $\mathbf{P} := (P, \leq)$ from a lattice $\mathbf{L} := (L, \wedge, \vee)$ define $x \leq y$ when $x = x \wedge y$.
- We can thus think of lattices either as posets or as algebras.
- In the case of $(\mathbb{N}, \leq)$ the corresponding algebra is $(\mathbb{N}, \min, \max)$.

Distributive lattices

Definition (Distributive lattice)

We say that a lattice **L** is *distributive* when **L** has for all $x, y, z \in L$ that

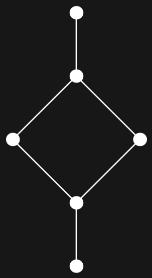
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

- The lattice $(\mathbb{N}, \leq)$ is distributive.
- It turns out that a lattice is distributive exactly when for all $x, y, z \in L$ we have

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

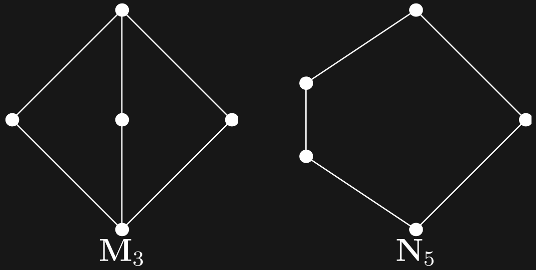
Distributive lattices

- This lattice is also distributive.



Distributive lattices

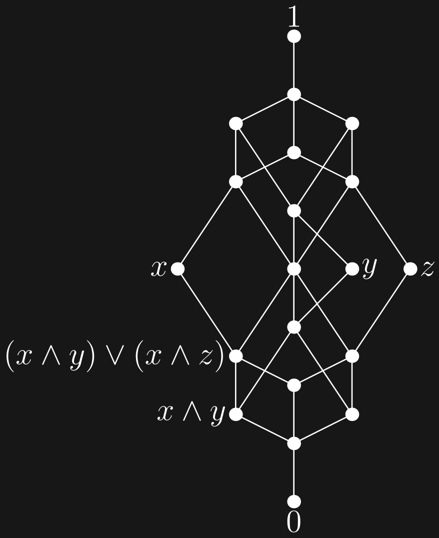
- There are exactly two nondistributive lattices of with 5 elements.



Free distributive lattices

- There are special distributive lattices, which are called *free distributive lattices*.
- I won't formally define these in this talk, but they are in a certain sense the most general possible distributive lattices.
- For each set there is a free distributive lattice generated by that set.
- I wanted to make an earring that looked like the Hasse diagram for the free distributive lattice on a set of three generators, say x, y, and z.
- We'll call this lattice $\mathbf{FD}_3$.

Free distributive lattices



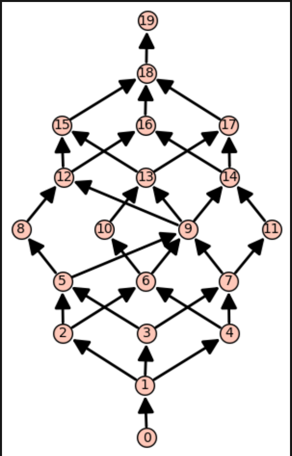
Drawing the earring

- I used the Sage computer algebra system to draw my earring.
- Sage can do calculations with posets and can draw Hasse diagrams

```
P = Poset((range(20),[[0,1],[1,2],[1,3],[1,4],[2,5]
[2,6],[3,5],[3,7],[4,6],[4,7],[5,8],[5,9],[6,9],
[6,10],[7,9],[7,11],[8,12],[9,12],[9,13],[9,14],
[10,13],[11,14],[12,15],[12,16],[13,15],[13,17],
[14,16],[14,17],[15,18],[16,18],[17,18],[18,19]]))
P.show()
```

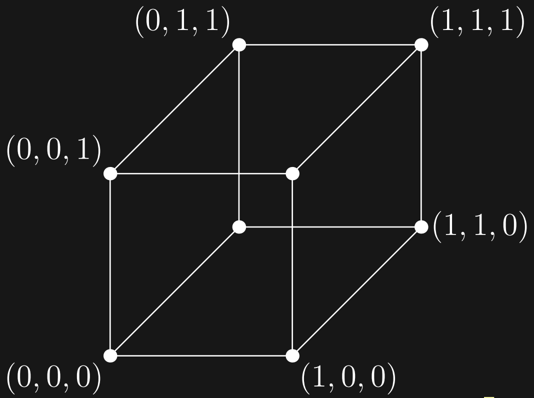
Drawing the earring

- Sage's Hasse diagram for $\mathbf{FD}_3$ looks good, but it's only 2D.



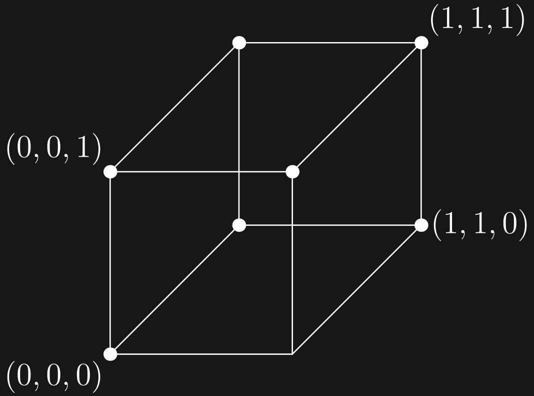
Drawing the earring

- I needed two copies of the unit cube pictured below, but I wanted the plane between them to be the *xy*-plane.
- I could accomplish this by rotating the cube so that $(0, 0, 0)$ stays fixed and $(1, 1, 1)$ goes to the positive *z*-axis.



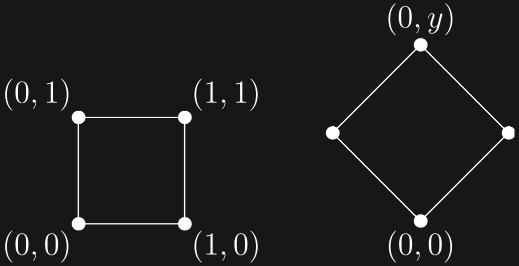
Drawing the earring

- This amounts to rotating the rectangle with vertices $(0,0,0)$, $(1,1,0)$, $(1,1,1)$, and $(0,0,1)$ about the origin in the plane it spans.



Drawing the earring

- First let's look at an analogous rotation of a square in the *xy*-plane.

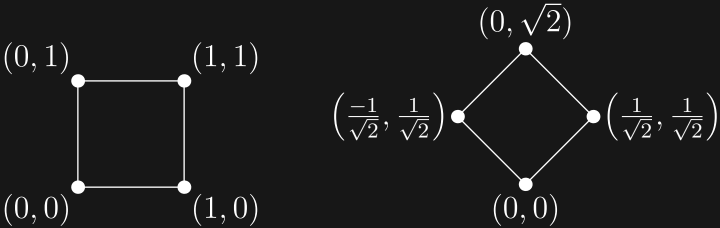


Drawing the earring

- Some trigonometry tells us this is a rotation counterclockwise by an angle of $\frac{\pi}{4}$ radians.
- A general rotation by an angle of θ can be represented by the matrix
$$r_\theta := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$
- Taking $\theta = \frac{\pi}{4}$ we have
$$r_{\frac{\pi}{4}} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Drawing the earring

- Multiplying by the matrix $r_{\frac{\pi}{4}}$ turns the original square into the one on the right.

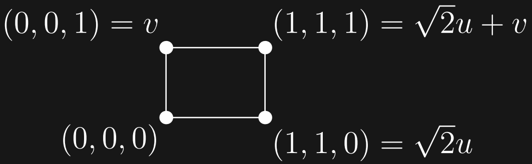


Drawing the earring

- Now we take

$$u := \frac{1}{\|(1,1,0)\|}(1,1,0) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

  and $v := (0,0,1)$.
- Consider the rectangle in the *uv*-plane.



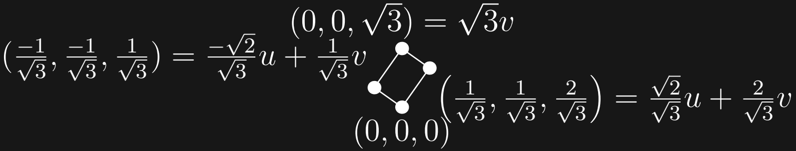
Drawing the earring

- Some trigonometry tells us this is a rotation counterclockwise by an angle θ with $\cos(\theta) = \frac{1}{\sqrt{3}}$ and $\sin(\theta) = \frac{\sqrt{2}}{\sqrt{3}}$.
- For this particular θ we have

$$r_\theta = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}.$$

Drawing the earring

- Multiplying by the matrix r_θ turns the original rectangle into the one below.



Drawing the earring

- A similar analysis for the other vertices of the cube gives the following vertices for the rotated cube.

```
a = 1/(2*sqrt(3))
b = 1/2
c = 1/sqrt(3)
cube0 = [(0,0,0),(1+a-b,a-b,c),(a-b,1+a-b,c),(-c,-c,c),
(c,c,2*c),(1-a-b,-a-b,2*c),(-a-b,1-a-b,2*c),(0,0,3*c)]
```

Drawing the earring

- By symmetry the other cube is obtained by negating *z*-coordinates.

```
a = 1/(2*sqrt(3))
b = 1/2
c = 1/sqrt(3)
cube0 = [(0,0,0),(1+a-b,a-b,c),(a-b,1+a-b,c),(-c,-c,c),
(c,c,2*c),(1-a-b,-a-b,2*c),(-a-b,1-a-b,2*c),(0,0,3*c)]
cube1 = [(0,0,0),(1+a-b,a-b,-c),(a-b,1+a-b,-c),
(-c,-c,-c),(c,c,-2*c),(1-a-b,-a-b,-2*c),
(-a-b,1-a-b,-2*c),(0,0,-3*c)]
```

Drawing the earring

- Now we just hard code in the edges of the Hasse diagram for $\mathbf{FD}_3$...

```
edges0 = [(cube0[0],cube0[1]),(cube0[0],cube0[2]),
(cube0[0],cube0[3]),(cube0[1],cube0[4]),(cube0[1],
cube0[5]),(cube0[2],cube0[4]),(cube0[2],cube0[6]),
(cube0[3],cube0[5]),(cube0[3],cube0[6]),
(cube0[4],cube0[7]),(cube0[5],cube0[7]),
(cube0[6],cube0[7])]
edges1 = [(cube1[0],cube1[1]),(cube1[0],cube1[2]),
(cube1[0],cube1[3]),(cube1[1],cube1[4]),(cube1[1],
cube1[5]),(cube1[2],cube1[4]),(cube1[2],cube1[6]),
(cube1[3],cube1[5]),(cube1[3],cube1[6]),
(cube1[4],cube1[7]),(cube1[5],cube1[7]),
(cube1[6],cube1[7])]
```

Drawing the earring

- Now we just hard code in the edges of the Hasse diagram for $\mathbf{FD}_3$...

```
edges2 = [(cube0[1],(2*(1+a-b),2*(a-b),0)),
(cube1[1],(2*(1+a-b),2*(a-b),0)),(cube0[2],
(2*(a-b),2*(1+a-b),0)),(cube1[2],
(2*(a-b),2*(1+a-b),0)),(cube0[3],
(-2*c,-2*c,0)),(cube1[3],(-2*c,-2*c,0))]
edges = edges0+edges1+edges2+
[((0,0,3*c),(0,0,3*c+1)),((0,0,-3*c),(0,0,-3*c-1))]
```

Drawing the earring

- Import Sage's 3D line segment and sphere constructors (if we're in the command line or an IDE)...

```
from sage.plot.plot3d.shapes import LineSegment,Sphere
```

Drawing the earring

- And assemble the earring with them!

```
L = None
for edge in edges:
    L += LineSegment(edge[0],edge[1],radius=0.1)
vertices = cube0+cube1[1:]+[(2*(1+a-b),2*(a-b),0),
(2*(a-b),2*(1+a-b),0),(-2*c,-2*c,0),(0,0,3*c+1),
(0,0,-3*c-1)]
for vert in vertices:
    L += Sphere(0.1).translate(vert)
L.show()
```

Printing the earring

- Sadly, 3D printing isn't totally trivial.
- I've been able to print other things on my printer, but this earring is too difficult.
- It's too fine, so I need to make it solid and bigger with more printing supports, but I need it to be small and light so it can be an earring.
- I could really use a metal laser sintering machine so I could make this thing from silver, gold, etc. in the right size, but those are still expensive. (Easily over USD$10000.)
- I still need to try the printers on campus too.

Thank you.