A High School Algebra Problem

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The Natural Numbers

- We denote by $\mathbb{N}$ the set of *natural numbers* $\mathbb{N} := \{1, 2, 3, \dots\}$.
- We can add $+$ multiply $\cdot$ pairs of natural numbers to obtain other natural numbers. We also often use the natural number 1 as a constant.
- The set $\mathbb{N}$ together with $+$, $\cdot$, and 1 forms the algebra $\widehat{\mathbf{N}}$ of the *restricted naturals* $\widehat{\mathbf{N}} := (\mathbb{N}, +, \cdot, 1)$.

Operations

- We denote by $\mathbb{W}$ the set of *whole numbers* $\mathbb{W} := \{0, 1, 2, \dots\}$.
- Given a set A and some $n \in \mathbb{W}$ we call a function $f\colon A^n \to A$ an *operation* and say that f is *n-ary*.
- We think of f as a «multiplication» on A, with the «product» of $a_1, a_2, \dots, a_n \in A$ (in that order) being $f(a_1, a_2, \dots, a_n)$.
- For example, $+$ and $\cdot$ on $\mathbb{N}$ are 2-ary (or *binary*) operations, while 1 can be viewed as a 0-ary (or *nullary*) operation on $\mathbb{N}$.

Abstract Algebras

- An *algebra* $\mathbf{A} := (A, f_1, f_2, \ldots, f_k)$ consists of a set A and a sequence of operations on A.
- The algebra of restricted naturals $\widehat{N} := (\mathbb{N}, +, \cdot, 1)$ is indeed an algebra according to our definition.
- When f_i is n_i-ary for each $1 \leq i \leq k$ we say that $\mathbf{A}$ has *signature* $(n_1, n_2, \ldots, n_k)$.
- We see that $\widehat{\mathbf{N}}$ has signature $(2, 2, 0)$.

Identities

- Given a signature $(n_1, n_2, \ldots, n_k)$ and operation symbols $f_1, f_2, \ldots, f_k$ an *identity* for $n_1, n_2, \ldots, n_k$ is a pair of expressions involving the operation symbols and some variables.
- The pair $(x + (y \cdot x), (y + 1) \cdot x)$ is an identity in the signature of $\widehat{\mathbf{N}}$. We usually write $x + (y \cdot x) \approx (y + 1) \cdot x$ instead.
- We say that $\mathbf{A} \models \epsilon$ when ϵ is an identity which is true for all assignments of the variables in ϵ.
- We have

  $$\widehat{\mathbf{N}} \models x + (y \cdot x) \approx (y + 1) \cdot x$$

  but

  $$\widehat{\mathbf{N}} \not\models x + y \approx (x \cdot y) + 1.$$

The Restricted High School Identities

There are six basic identities which are modeled by $\widehat{\mathbf{N}}$. We call this collection of identities the *restricted high school identities* $\widehat{\mathrm{HSI}}$.

1. $x + y \approx y + x$
2. $x + (y + z) \approx (x + y) + z$
3. $x \cdot 1 \approx x$
4. $x \cdot y \approx y \cdot x$
5. $x \cdot (y \cdot z) \approx (x \cdot y) \cdot z$
6. $x \cdot (y + z) \approx (x \cdot y) + (x \cdot z)$

Polynomial normal form

- Using $\widehat{\mathrm{HSI}}$ we can convert any expression in the signature of $\widehat{\mathbf{N}}$ into a polynomial.
- For example, $(x \cdot (y + (x + 1))) + x$ yields $xy + x^2 + 2x$.
- An identity $\alpha \approx \beta$ is true in $\widehat{\mathbf{N}}$ exactly when the polynomials for α and β are the same.
- This means that there is an algorithm for checking whether an identity holds in $\widehat{\mathbf{N}}$ and that $\widehat{\mathrm{HSI}}$ can be used to derive any true identity of $\widehat{\mathbf{N}}$.

Exponentiation

- We can also exponentiate $\uparrow$ pairs of natural numbers to obtain other natural numbers.
- Including $\uparrow$ we have the algebra **N** of the *naturals* $\mathbf{N} := (\mathbb{N}, +, \cdot, \uparrow, 1)$.
- The signature of **N** is $(2, 2, 2, 0)$.

Dedekind's High School Identities

In 1888 Richard Dedekind gave a list of basic identities which are modeled by **N**. They consist of $\widehat{\mathrm{HSI}}$ as well as the following five identities involving exponentiation. We call this collection of identities the *high school identities* HSI.

1. $1^x \approx 1$
2. $x^1 \approx x$
3. $x^{y+z} \approx x^y \cdot x^z$
4. $(x \cdot y)^z \approx x^z \cdot y^z$
5. $(x^y)^z \approx x^{y \cdot z}$

Tarski's High School Algebra Problem

- Since $\widehat{\mathrm{HSI}}$ can be used to derive every true identity in $\hat{\mathbf{N}}$ it is natural to ask whether every true identity in $\mathbf{N}$ follows from HSI.
- This is the question that Alfred Tarski asked in the 1960s, which we call *Tarski's High School Algebra Problem*.
- This is equivalent to knowing whether there exist *exotic identities*, identities ϵ for which $\mathbf{N} \models \epsilon$ but which cannot be proven using HSI.

Wilkie's Exotic Identity

- It turns out that exotic identities exist.
- In the early 1980s Alex Wilkie produced the first example of an exotic identity for **N**, solving Tarski's High School Algebra Problem.
- This *Wilkie identity* $W(x, y)$ is

$$((1+x)^y + (1+x+x^2)^y)^x \cdot ((1+x^3)^x + (1+x^2+x^4)^x)^y$$
$$\approx ((1+x)^x + (1+x+x^2)^x)^y \cdot ((1+x^3)^y + (1+x^2+x^4)^y)^x.$$

- The proof that $\mathbf{N} \models W(x, y)$ uses analysis on the corresponding functions on the real numbers $\mathbb{R}$.
- Wilkie's proof that HSI does not imply $W(x, y)$ was somewhat abstract, but in 1985 Gurevič gave a more concrete proof.
- Imagine we had an algebra **A** such that $\mathbf{A} \models \mathrm{HSI}$. If $\mathbf{A} \not\models W(x, y)$ then we can't have that HSI implies $W(x, y)$, for otherwise we would be able to prove $W(x, y)$ for **A**.
- Gurevič found such an algebra **A** with $|A| = 59$.

A Small Counterexample to Tarski's Problem

We give an algebra $\mathbf{A} := (A, +, \cdot, \uparrow, 1)$ with $|A| = 12$ which was found in 2001 by Burris and Yeats.

A Small Counterexample to Tarski's Problem

+	1	2	3	4	a	b	c	d	e	f	g	h
1	2	3	4	4	2	3	d	3	3	3	3	4
2	3	4	4	4	3	4	3	4	4	4	4	4
3	4	4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4	4	4
a	2	3	4	4	b	4	b	3	h	3	3	4
b	3	4	4	4	4	4	4	4	4	4	4	4
c	d	3	4	4	b	4	b	3	3	3	3	4
d	3	4	4	4	3	4	3	4	4	4	4	4
e	3	4	4	4	h	4	3	4	4	3	h	4
f	3	4	4	4	3	4	3	4	3	4	3	4
g	3	4	4	4	3	4	3	4	h	3	4	4
h	4	4	4	4	4	4	4	4	4	4	4	4

A Small Counterexample to Tarski's Problem

×	1	2	3	4	a	b	c	d	e	f	g	h
1	1	2	3	4	a	b	c	d	e	f	g	h
2	2	4	4	4	b	4	b	4	4	4	4	4
3	3	4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4	4	4
a	a	b	4	4	c	b	c	b	h	4	4	4
b	b	4	4	4	b	4	b	4	4	4	4	4
c	c	b	4	4	c	b	c	b	4	4	4	4
d	d	4	4	4	b	4	b	4	4	4	4	4
e	e	4	4	4	h	4	4	4	4	4	h	4
f	f	4	4	4	4	4	4	4	4	4	4	4
g	g	4	4	4	4	4	4	4	h	4	4	4
h	h	4	4	4	4	4	4	4	4	4	4	4

A Small Counterexample to Tarski's Problem

$\uparrow$	1	2	3	4	a	b	c	d	e	f	g	h
1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	4	4	4	4	4	4	4	f	4	4	4
3	3	4	4	4	e	4	4	4	g	4	e	h
4	4	4	4	4	4	4	4	4	4	4	4	4
a	a	c	c	c	c	c	c	c	c	c	c	c
b	b	4	4	4	4	4	4	4	4	4	4	4
c	c	c	c	c	c	c	c	c	c	c	c	c
d	d	4	4	4	f	4	4	4	4	4	4	4
e	e	4	4	4	4	4	4	4	h	4	4	4
f	f	4	4	4	4	4	4	4	4	4	4	4
g	g	4	4	4	h	4	4	4	4	4	h	4
h	h	4	4	4	4	4	4	4	4	4	4	4

A Longer List of Basic Identities

- One might wonder whether adding $W(x, y)$ to HSI would make a new list of identities from which every true identity in **N** can be proven.
- This is not the case.
- In 1990 Gurevič showed that there cannot be any finite list of such identities.

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Thank you.