Orientable smooth manifolds are essentially quasigroups

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Introduction

- In the mid-2010s Herman and Pakianathan introduced a functorial construction of closed surfaces from noncommutative finite groups.
- Semin Yoo and I decided to produce an n-dimensional generalization.
- The two main challenges in doing this were finding an appropriate analogue of noncommutative groups and in desingularizing the n-dimensional pseudomanifolds which arose at the first stage of our construction.
- Ultimately we found that every orientable triangulable manifold could be manufactured in the manner we described.

Talk outline

- Herman and Pakianathan's construction
- Quasigroups
- The first functor: Open serenation
- The second functor: Serenation
- The Evans Conjecture and Latin cubes

Herman and Pakianathan's construction

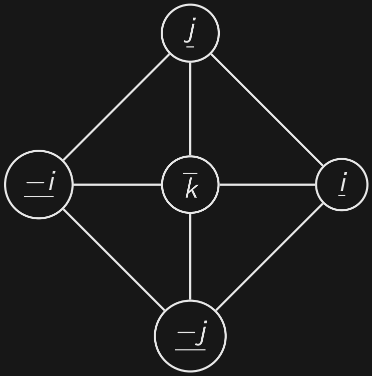
- Consider the quaternion group **G** of order 8 whose universe is $G := \{\pm 1, \pm i, \pm j, \pm k\}$.
- We begin by picking out all the pairs of elements $(x, y) \in G^2$ so that $xy \neq yx$. We call this collection $\mathrm{NCT}(\mathbf{G})$.
- We define $\mathrm{In}(\mathbf{G})$ to be all the elements of G which are entries in some pair $(x, y) \in \mathrm{NCT}(\mathbf{G})$.
- Similarly, $\mathrm{Out}(\mathbf{G})$ is defined to be all the members of G of the form xy where $(x, y) \in \mathrm{NCT}(\mathbf{G})$.

Herman and Pakianathan's construction

- In this case we have
$$\mathrm{NCT}(\mathbf{G}) = \left\{ (\pm u, \pm v) \;\middle|\; \{u, v\} \in \binom{\{i, j, k\}}{2} \right\}$$
so
$$\mathrm{In}(\mathbf{G}) = \{\pm i, \pm j, \pm k\}$$
and
$$\mathrm{Out}(\mathbf{G}) = \{\pm i, \pm j, \pm k\}\,.$$
- From this data we form a simplicial complex (actually a 2-pseudomanifold) whose facets are of the form $\{\underline{x}, \underline{y}, \overline{xy}\}$ where $(x, y) \in \mathrm{NCT}(\mathbf{G})$.

Herman and Pakianathan's construction

- One «sheet» of this complex is pictured below.

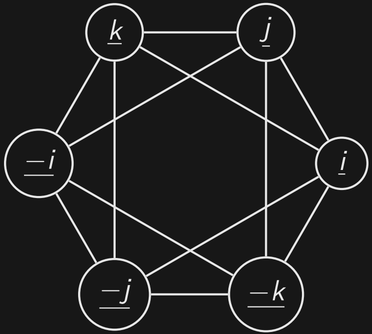


Herman and Pakianathan's construction

- The three 4-cycles

$$(\underline{i}, \underline{j}, \underline{-i}, \underline{-j}),\ (\underline{i}, \underline{k}, \underline{-i}, \underline{-k}),\ \text{and}\ (\underline{j}, \underline{k}, \underline{-j}, \underline{-k}).$$

each carry an octohedron.



Herman and Pakianathan's construction

- This simplicial complex, which we call **Sim(G)** and Herman and Pakianathan called $X(Q_8)$, consists of three 2-spheres, each pair of which is glued at two points.
- Deleting these points to disjointize the spheres and filling the resulting holes yields the manifold we call **Ser(G)** and Herman and Pakianathan called $Y(Q_8)$.
- In this case **Ser(G)** is the disjoint union of three 2-spheres.

Quasigroups

Definition (Quasigroup)

A *(binary) quasigroup* is a magma $\mathbf{A} := (A, f\colon A^2 \to A)$ such that if any two of the variables x, y, and z are fixed the equation

$$f(x, y) = z$$

has a unique solution.

- That is, a quasigroup is a magma whose Cayley table is a Latin square, where each entry occurs once in each row and each column.
- All groups are quasigroups, but quasigroups need not have identities or be associative.

Quasigroups

- The midpoint operation

$$f(x, y) := \frac{1}{2}(x + y)$$

  is a quasigroup operation on $\mathbb{R}^n$.
- The magma $(\mathbb{Z}, -)$ is a quasigroup.

Quasigroups

Definition (Quasigroup)

A *(binary) quasigroup* is an algebra $\mathbf{A} := (A, f, g_1, g_2)$ where for all $x_1, x_2, y \in A$ we have

$$f(g_1(x_2, y), x_2) = y,$$

$$f(x_1, g_2(x_1, y)) = y,$$

$$g_1(x_2, f(x_1, x_2)) = x_1,$$

and

$$g_2(x_1, f(x_1, x_2)) = x_2.$$

- We think of $g_1(x, y)$ as the division of y by x in the second coordinate.

Quasigroups

- The preceding definition shows that the class Quas_2 of all binary quasigroups can be defined by universally-quantified equations, or *identities*.
- This means that Quas_2 is a variety of algebras in the sense of universal algebra, and hence forms a category $\mathbf{Quas}_2$ which is closed under taking quotients, subalgebras, and products.
- Note that Herman and Pakianathan's construction works with noncommutative quasigroups just as well as with groups.
- We would then like an n-ary version of a quasigroup for our n-dimensional generalization.

Quasigroups

Definition (Quasigroup)

An *n-quasigroup* is an n-magma $\mathbf{A} := (A, f\colon A^n \to A)$ such that if any $n-1$ of the variables $x_1, \ldots, x_n, y$ are fixed the equation

$$f(x_1, \ldots, x_n) = y$$

has a unique solution.

- That is, an n-quasigroup is an n-magma whose Cayley table is a Latin n-cube.
- All n-ary groups are quasigroups, but quasigroups need not be associative.

Quasigroups

- Given any group **G** the *n*-ary multiplication

$$f(x_1, \ldots, x_n) := x_1 \cdots x_n$$

is a quasigroup operation on G.

Quasigroups

Definition (Quasigroup)

An *n-quasigroup* is an algebra

$$\mathbf{A} := (A, f, g_1, \ldots, g_n)$$

where for all $x_1, \ldots, x_n, y \in A$ and each $i \in \{1, 2, \ldots, n\}$ we have

$$f(x_1, \ldots, x_{i-1}, g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y), x_{i+1}, \ldots, x_n) = y$$

and

$$g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, f(x_1, \ldots, x_n)) = x_i.$$

- We think of $g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y)$ as the division of y simultaneously by x_j in the j^{th} coordinate for each $j \neq i$.

Quasigroups

- We say that an n-quasigroup $\mathbf{A}$ is *commutative* when for all $x_1, \ldots, x_n \in A$ and all $\sigma \in \mathrm{Perm}_n$ we have

$$f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

- We say that an n-quasigroup $\mathbf{A}$ is *alternating* when for all $x_1, \ldots, x_n \in A$ and all $\sigma \in \mathrm{Alt}_n$ we have

$$f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

- Our "correct" analogue of the variety of groups will be the variety AQ_n of alternating n-ary quasigroups.

Quasigroups

- There are nontrivial members of AQ_n for each n, but the easiest examples are either commutative (take the n-ary multiplication for an abelian group) or infinite (the free alternating quasigroups, which we need later but are too much right now).
- We tediously found the following example by hand:

Quasigroups

- Take $S := (\mathbb{Z}/5\mathbb{Z})^3$ and define $h\colon \mathbb{Z}/5\mathbb{Z} \times \mathbf{Alt}_3 \to \mathbf{Perm}_S$ by

  $$(h(k,\sigma))(x_1, x_2, x_3) := (x_{\sigma(1)} + k, x_{\sigma(2)} + k, x_{\sigma(3)} + k).$$

  There are 7 members of $\mathrm{Orb}(h)$. One system of orbit representatives is:

  $$\{000, 011, 022, 012, 021, 013, 031\}\,.$$

Quasigroups

- Let $A := \mathbb{Z}/5\mathbb{Z}$ and define a ternary operation $f\colon A^3 \to A$ so that
$$f((h(k,\sigma))(x_1,x_2,x_3)) = f(x_1,x_2,x_3) + k$$
and f is defined on the above set of orbit representatives as follows.

xyz	$f(x,y,z)$
000	0
011	0
022	0
012	3
021	4
013	4
031	2

Quasigroups

- By taking products of $\mathbf{A} := (A, f)$ this gives us infinitely many finite, noncommutative, alternating ternary quasigroups, but we only have one basic example.
- We reached out to Jonathan Smith to see if anyone had studied the varieties of alternating n-quasigroups before, but it seemed that no one had.
- He did, however, give us an example which we generalized into an *alternating product* construction which takes an n-ary commutative quasigroup and an $(n+1)$-ary commutative quasigroup and yields an n-ary alternating quasigroup which is typically not commutative.

Quasigroups

Definition (Alternating map)

Given sets A and B we say that a function $\alpha\colon A^n \to B$ is an *n-ary alternating map* from A to B when for each $\sigma \in \mathrm{Alt}_n$ and each $a \in A^n$ we have that

$$\alpha(a) = \alpha(a_{\sigma(1)}, \dots, a_{\sigma(n)}).$$

- Note that the determinant is an alternating n-ary map from $\mathbb{F}^n$ to $\mathbb{F}$ for any field $\mathbb{F}$.

Quasigroups

Definition (Alternating product)

Given an n-ary commutative quasigroup $\mathbf{U} := (U, g)$, an $(n+1)$-ary commutative quasigroup $\mathbf{V} := (V, h)$, and an n-ary alternating map $\alpha\colon A^n \to B$ the *alternating product* of $\mathbf{U}$ and $\mathbf{V}$ with alternating map α is the alternating n-quasigroup

$$\mathbf{U} \boxtimes_\alpha \mathbf{V} := (U \times V, g \boxtimes_\alpha h\colon (U \times V)^n \to U \times V)$$

where for $(u_1, v_1), \ldots, (u_n, v_n) \in U \times V$ we define

$$(g \boxtimes_\alpha h)((u_1, v_1), \ldots, (u_n, v_n)) := (g(u), h(\alpha(u), v_1, \ldots, v_n))$$

where $u := (u_1, \ldots, u_n)$.

Quasigroups

- The variety of n-quasigroups (not necessarily alternating) is congruence permutable, and hence congruence modular.
- Note the similarity between the alternating product $\mathbf{U} \boxtimes_\alpha \mathbf{V}$ and the decomposition decomposition of an algebra $\mathbf{A}$ in a congruence modular variety as $\mathbf{Q} \otimes^T \mathbf{B}$ where $\mathbf{Q}$ is Abelian and $\mathbf{B} := \mathbf{A}/\zeta_\mathbf{A}$.
- Note also the similarity between this construction and the factor set construction of group extensions with an abelian kernel.

Quasigroups

Definition (Commuting tuple)

Given $\mathbf{A} := (A, f) \in \mathrm{AQ}_n$ we say that $a \in A^n$ *commutes* (or is a *commuting tuple*) in $\mathbf{A}$ when we have for each $\sigma \in \mathrm{Perm}_n$ that

$$f(a) = f(a_{\sigma(1)}, \ldots, a_{\sigma(n)}).$$

Definition (Set of noncommuting tuples)

Given $\mathbf{A} := (A, f) \in \mathrm{AQ}_n$ we define the *noncommuting tuples* $\mathrm{NCT}(\mathbf{A})$ of $\mathbf{A}$ by

$$\mathrm{NCT}(\mathbf{A}) := \{\, a \in A^n \mid a \text{ does not commute in } \mathbf{A} \,\}.$$

Quasigroups

Definition (NC homomorphism)

We say that a homomorphism $h\colon \mathbf{A}_1 \to \mathbf{A}_2$ of alternating quasigroups is an *NC homomorphism* (or a *noncommuting homomorphism*) when for each $a \in \mathrm{NCT}(\mathbf{A}_1)$ we have that

$$h(a) = (h(a_1), \ldots, h(a_n)) \in \mathrm{NCT}(\mathbf{A}_2).$$

- It's tempting to say that the NC congruences of $\mathbf{A}$ should be those contained in the center of $\mathbf{A}$ but we aren't sure whether that is always the case yet.

The first functor: Open serenation

- Our first construction gives a functor

$$\mathbf{OSer}_n \colon \mathbf{NCAQ}_n \to \mathbf{SMfld}_n\,.$$

- We define

$$\mathbf{Sim}_n \colon \mathbf{NCAQ}_n \to \mathbf{PMfld}_n$$

  similarly to our previous example for $n = 2$.
- We define $\mathrm{In}(\mathbf{A})$ to consist of all entries in noncommuting tuples of $\mathbf{A}$ and $\mathrm{Out}(\mathbf{A})$ to consist of all $f(a_1, \ldots, a_n)$ where $(a_1, \ldots, a_n) \in \mathrm{NCT}(\mathbf{A})$.

The first functor: Open serenation

- We set

$$\mathrm{Sim}(\mathbf{A}) := \{\, \underline{a} \mid a \in \mathbf{In}(\mathbf{A}) \,\} \cup \{\, \overline{a} \mid a \in \mathrm{Out}(\mathbf{A}) \,\}$$

  and

$$\mathrm{SimFace}(\mathbf{A}) := \bigcup_{a \in \mathrm{NCT}(\mathbf{A})} \mathrm{Sb}\left(\left\{\underline{a}_1, \dots, \underline{a}_n, \overline{f(a)}\right\}\right).$$

- We define

$$\mathbf{Sim}_n(\mathbf{A}) := (\mathrm{Sim}(\mathbf{A}), \mathrm{SimFace}(\mathbf{A})).$$

The first functor: Open serenation

- We create $\mathbf{OSer}_n(\mathbf{A})$ by taking the open geometric realization of $\mathbf{Sim}_n(\mathbf{A})$ (basically all but the $(n-2)$-skeleton of the open geometric realization) and then equipping it with a smooth atlas.
- The *standard open bipyramid* (or just *bipyramid*) in $\mathbb{R}^n$ is

$$\mathrm{Bipyr}_n := \mathrm{OCvx}\left(\left\{(0,\ldots,0), \left(\frac{2}{n},\ldots,\frac{2}{n}\right)\right\} \cup \{e_1,\ldots,e_n\}\right)$$

where e_i is the i^{th} standard basis vector of $\mathbb{R}^n$.

The first functor: Open serenation

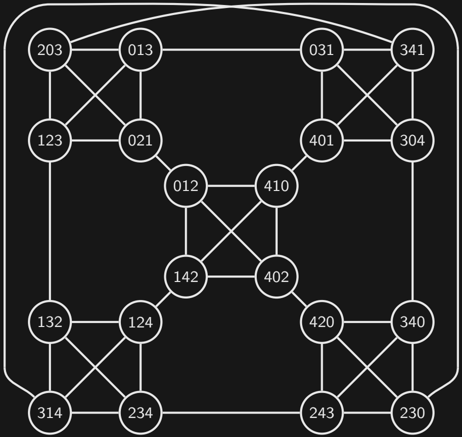
- Given an alternating n-quasigroup $\mathbf{A}$ and $a = (a_1, \ldots, a_n) \in \mathsf{NCT}(\mathbf{A})$ the *serene chart of input type* for a is
$$\underline{\phi}_a \colon \mathrm{Bipyr}_n \to \mathrm{OSer}_n(\mathbf{A}).$$
- We set
$$\underline{\phi}_a(u_1, \ldots, u_n) := \sum_{i=1}^{n} u_i \underline{a}_i + \left(1 - \sum_{i=1}^{n} u_i\right) \overline{f(a)}$$
when $\sum_{i=1}^{n} u_i \leq 1$.
- Otherwise,
$$\underline{\phi}_a(u_1, \ldots, u_n) := \frac{2}{n} \sum_{i=1}^{n} \left(1 + \frac{n-2}{2} u_i - \sum_{j \neq i} u_j\right) \underline{a}_i + \left(-1 + \sum_{i=1}^{n} u_i\right) \overline{f(a')}.$$

The first functor: Open serenation

- There are also serene charts of output type, where are defined similarly.
- We set
$$(\mathrm{OSer}_n(\mathbf{A}), \tau) := (\mathbf{OGeo}_n \circ \mathbf{Sim}_n)(\mathbf{A}).$$
- We then define
$$\mathbf{OSer}_n(\mathbf{A}) := (\mathrm{OSer}_n(\mathbf{A}), \tau, \mathrm{SerAt}_n(\mathbf{A}))$$
where
$$\mathrm{SerAt}_n(\mathbf{A}) := \bigcup_{a \in \mathrm{NCT}(\mathbf{A})} \left\{ \underline{\phi}_a, \overline{\phi}_a \right\}.$$

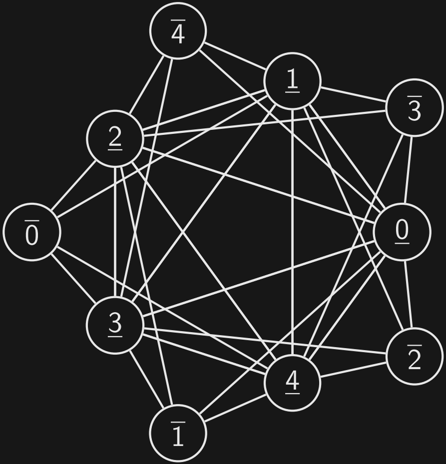
The first functor: Open serenation

- The incidence graph of the facets of **Sim**(**A**) for the ternary quasigroup **A** from the previous example is pictured.



The first functor: Open serenation

- The 1-skeleton of **Sim**(**A**) for the ternary quasigroup **A** from the previous example is pictured.



The first functor: Open serenation

- One may verify that **OSer**(**A**) is a 3-sphere minus the graph pictured previously, which is homotopy equivalent to the wedge sum of 21 circles.

The second functor: Serenation

- For any alternating quasigroup $\mathbf{A}$ we may equip $\mathbf{OSer}(\mathbf{A})$ with a Riemannian metric in a functorial manner which makes $\mathbf{OSer}(\mathbf{A})$ flat.
- We then define a *Euclidean metric completion functor*

$$\mathbf{EuCmplt}\colon \mathbf{Riem}_n \to \mathbf{Mfld}_n$$

which assigns to a Riemannian manifold $(\mathbf{M}, g)$ the topological manifold consisting of all points in the metric completion of $\mathbf{M}$ which are locally Euclidean.

The second functor: Serenation

- The *serenation functor*

$$\mathbf{Ser}_n\colon \mathbf{NCAQ}_n \to \mathbf{Mfld}_n$$

  is given by

$$\mathbf{Ser}(\mathbf{A}) := \mathbf{EuCmplt}(\mathbf{OSer}(\mathbf{A}), g)$$

  where g is the standard metric on $\mathbf{OSer}(\mathbf{A})$.
- In the previous example of the ternary quasigroup $\mathbf{A}$ we find that $\mathbf{Ser}_3(\mathbf{A})$ is the 3-sphere.

The second functor: Serenation

Definition (Serene manifold)

We say that a connected orientable n-manifold $\mathbf{M}$ is *serene* when there exists some alternating n-quasigroup $\mathbf{A}$ such that $\mathbf{M}$ is a component of $\mathbf{Ser}(\mathbf{A})$.

The second functor: Serenation

Theorem (A., Yoo (2021))

Every connected orientable triangulable n-manifold is serene.

The second functor: Serenation

Theorem (A., Yoo (2021))

Every connected orientable triangulable n-manifold is serene.

- Consider a triangulation of the given manifold **M**.
- Subdivide each facet in a manner I will draw off to the side.
- We have that **M** is homeomorphic to a corresponding component of the serenation of a quotient of the free alternating n-quasigroup whose generators are the vertices of the subdivided triangulation.

The Evans Conjecture and Latin cubes

Definition (Quasifinite manifold)

We say that a connected compact orientable smooth n-manifold $\mathbf{M}$ is *quasifinite* when there exists a finite alternating n-quasigroup $\mathbf{A}$ such that $\mathbf{M}$ is homeomorphic to a component of $\mathbf{Ser}(\mathbf{A})$.

- Is every connected compact orientable smooth manifold quasifinite?

The Evans Conjecture and Latin cubes

Definition (Partial Latin cube)

Given a set A and some $n \in \mathbb{N}$ we say that $\theta \subset A^{n+1}$ is a *partial Latin n-cube* when for each $i \in [n]$ and each

$$a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n+1} \in A^n$$

there exists at most one $a_i \in A$ so that

$$(a_1, \ldots, a_{n+1}) \in \theta.$$

The Evans Conjecture and Latin cubes

- Evans conjectured that each partial Latin square (i.e. a partial Latin cube $\theta \subset A^{2+1}$) with $|A| = k$ and $|\theta| \leq n - 1$ could be filled in so as to obtain a complete Latin square $\psi \subset A^3$ with $\theta \subset \psi$ and $|\psi| = k^2$.
- This was proven to be true by Smetaniuk in 1981.
- Similar results are known for special classes of higher-dimensional Latin cubes.

The Evans Conjecture and Latin cubes

- In general a *complete Latin n-cube* is the graph of an n-quasigroup operation.
- We say that a partial Latin n-cube is *alternating* when we have for each $\alpha \in \mathrm{Alt}_n$ that if

$$(a_1, \ldots, a_n, b_1) \in \theta$$

  and

$$(a_{\alpha(1)}, \ldots, a_{\alpha(n)}, b_2) \in \theta$$

  then $b_1 = b_2$.
- Given a finite partial alternating Latin cube $\theta \subset A^{n+1}$ does there always exist a finite complete alternating Latin cube $\psi \subset B^{n+1}$ such that $\theta \subset \psi$?

The Evans Conjecture and Latin cubes

- We don't ask for any particular relationship between $|\theta|$ and $|B|$, so this is in one sense a weaker question than the Evans Conjecture. That is, we may add many new elements to A in order to complete our Latin cube, as long as we only add finitely many.
- We have a corollary of the Evans Conjecture for the $n = 2$ case.

Corollary

Every connected compact orientable surface is a component of the serenation of some finite binary quasigroup.

References

- Mark Herman and Jonathan Pakianathan. "On a canonical construction of tessellated surfaces from finite groups". In: *Topology Appl.* 228 (2017), pp. 158–207. ISSN: 0166-8641