More Multiplayer Rock-Paper-Scissors

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RPS as a Magma

We will view the game of RPS as a magma. We let $A := \{r, p, s\}$ and define a binary operation $f: A^2 \to A$ where $f(x, y)$ is the winning item among $\{x, y\}$.

	r	p	s
r	r	p	r
p	p	p	s
s	r	s	s

Selection Games

A *selection game* is a game consisting of a collection of items A, from which a fixed number of players n each choose one, resulting in a tuple $a \in A^n$, following which the round's winners are those who chose $f(a)$ for some fixed rule $f\colon A^n \to A$. RPS is a selection game, and we can identify each such game with an *n-ary magma* $\mathbf{A} := (A, f)$.

Properties of RPS

The game RPS is

1. conservative,
2. essentially polyadic,
3. strongly fair, and
4. nondegenerate.

These are the properties we want for a multiplayer game, as well.

Properties of RPS: Conservativity

We say that an operation $f\colon A^n \to A$ is *conservative* when for any $a_1, \ldots, a_n \in A$ we have that $f(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$. We say that $\mathbf{A}$ is conservative when each round has at least one winning player.

Properties of RPS: Essential Polyadicity

We say that an operation $f\colon A^n \to A$ is *essentially polyadic* when there exists some $g\colon \mathrm{Sb}(A) \to A$ such that for any $a_1, \ldots, a_n \in A$ we have $f(a_1, \ldots, a_n) = g(\{a_1, \ldots, a_n\})$. We say that **A** is essentially polyadic when a round's winning item is determined solely by which items were played, not taking into account which player played which item or how many players chose a particular item.

Properties of RPS: Strong Fairness

Let A_k denote the members of A^n which have k distinct components for some $k \in \mathbb{N}$. We say that f is *strongly fair* when for all $a, b \in A$ and all $k \in \mathbb{N}$ we have $\left|f^{-1}(a) \cap A_k\right| = \left|f^{-1}(b) \cap A_k\right|$. We say that **A** is strongly fair when each item has the same chance of being the winning item when exactly k distinct items are chosen for any $k \in \mathbb{N}$.

Properties of RPS: Nondegeneracy

We say that f is *nondegenerate* when $|A| > n$. In the case that $|A| \leq n$ we have that all members of $A_{|A|}$ have the same set of components. If **A** is essentially polyadic with $|A| \leq n$ it is impossible for **A** to be strongly fair unless $|A| = 1$.

Variants with More Items

The French version of RPS adds one more item: the well. This game is not strongly fair but is conservative and essentially polyadic. The recent variant Rock-Paper-Scissors-Spock-Lizard is conservative, essentially polyadic, strongly fair, and nondegenerate.

	r	p	s	w
r	r	p	r	w
p	p	p	s	p
s	r	s	s	w
w	w	p	w	w

	r	p	s	v	l
r	r	p	r	v	r
p	p	p	s	p	l
s	r	s	s	v	s
v	v	p	v	v	l
l	r	l	s	l	l

Result for Two-Player Games

The only "valid" RPS variants for two players use an odd number of items.

Proposition

Let $\mathbf{A}$ *be a selection game with* $n = 2$ *which is essentially polyadic, strongly fair, and nondegenerate and let* $m := |A|$*. We have that* $m \neq 1$ *is odd. Conversely, for each odd* $m \neq 1$ *there exists such a selection game.*

PRPS Magmas

Definition (PRPS magma)

Let $\mathbf{A} := (A, f)$ be an n-ary magma. When $\mathbf{A}$ is essentially polyadic, strongly fair, and nondegenerate we say that $\mathbf{A}$ is a PRPS *magma* (read "*pseudo*-RPS *magma*"). When $\mathbf{A}$ is an n-magma of order $m \in \mathbb{N}$ with these properties we say that $\mathbf{A}$ is a PRPS(m, n) *magma*. We also use PRPS and PRPS(m, n) to indicate the classes of such magmas.

Result for Multiplayer Games

Theorem

Let $\mathbf{A} \in \mathrm{PRPS}(m,n)$ *and let* $\varpi(m)$ *denote the least prime dividing* m*. We have that* $n < \varpi(m)$*. Conversely, for each pair* (m,n) *with* $m \neq 1$ *such that* $n < \varpi(m)$ *there exists such a magma.*

RPS Magmas

Definition (RPS magma)

Let $\mathbf{A} := (A, f)$ be an n-ary magma. When $\mathbf{A}$ is conservative, essentially polyadic, strongly fair, and nondegenerate we say that $\mathbf{A}$ is an RPS *magma*. When $\mathbf{A}$ is an n-magma of order m with these properties we say that $\mathbf{A}$ is an $\mathrm{RPS}(m, n)$ *magma*. We also use RPS and $\mathrm{RPS}(m, n)$ to indicate the classes of such magmas.

α-action Magmas

Definition (α-action magma)

Fix a group $\mathbf{G}$, a set A, and some $n < |A|$. Given a regular group action $\alpha\colon \mathbf{G} \to \mathbf{Perm}(A)$ such that each of the k-extensions of α is free for $1 \leq k \leq n$ let $\Psi_k := \left\{ \operatorname{Orb}(U) \;\middle|\; U \in \binom{A}{k} \right\}$ where $\operatorname{Orb}(U)$ is the orbit of U under α_k. Let $\beta := \{\beta_k\}_{1\leq k\leq n}$ be a sequence of choice functions $\beta_k\colon \Psi_k \to \binom{A}{k}$ such that $\beta_k(\psi) \in \psi$ for each $\psi \in \Psi_k$. Let $\gamma := \{\gamma_k\}_{1\leq k\leq n}$ be a sequence of functions $\gamma_k\colon \Psi_k \to A$ such that $\gamma_k(\psi) \in \beta_k(\psi)$ for each $\psi \in \Psi_k$. Let $g\colon \mathrm{Sb}_{\leq n}(A) \to A$ be given by $g(U) := (\alpha(s))(\gamma_k(\psi))$ when $U = (\alpha_k(s))(\beta_k(\psi))$. Define $f\colon A^n \to A$ by $f(a_1, \ldots, a_n) := g(\{a_1, \ldots, a_n\})$. The *$\alpha$-action magma* induced by (β, γ) is $\mathbf{A} := (A, f)$.

α-action Magmas are RPS Magmas

Theorem

Let $\mathbf{A}$ *be an* α*-action magma induced by* (β, γ)*. We have that* $\mathbf{A} \in \mathrm{RPS}$*.*

Definition (Regular RPS magma)

Let $\mathbf{G}$ be a nontrivial finite group and fix $n < \varpi(|G|)$. We denote by $\mathbf{G}_n(\beta, \gamma)$ the L-action n-magma induced by (β, γ), which we refer to as a *regular* RPS *magma*.

A Game for Three Players

0	0	1	2	3	4
0	0	1	0	3	0
1	1	1	0	0	4
2	0	0	0	2	4
3	3	0	2	3	3
4	0	4	4	3	0

1	0	1	2	3	4
0	1	1	0	0	4
1	1	1	2	1	4
2	0	2	2	1	1
3	0	1	1	1	3
4	4	4	1	3	4

2	0	1	2	3	4
0	0	0	0	2	4
1	0	2	2	1	1
2	0	2	2	3	2
3	2	1	3	3	2
4	4	1	2	2	2

3	0	1	2	3	4
0	3	0	2	3	3
1	0	1	1	1	3
2	2	1	3	3	2
3	3	1	3	3	4
4	3	3	2	4	4

4	0	1	2	3	4
0	0	4	4	3	0
1	4	4	1	3	4
2	4	1	2	2	2
3	3	3	2	4	4
4	0	4	2	4	4

Functions Exhibiting Essential Polyadicity

U	0	1	2	01	12	20
$g(U)$	0	1	2	0	1	2

RPS

U	0	1	2	01	12	23	34	40	02	13	24	30	41
$g(U)$	0	1	2	1	2	3	4	0	0	1	2	3	4

U	012	123	234	340	401	013	124	230	341	402
$g(U)$	0	1	2	3	4	0	1	2	3	4

RPS(5, 3) example

Hypergraphs

Definition (Pointed hypergraph)

A *pointed hypergraph* $\mathbf{S} := (S, \sigma, g)$ consists of a hypergraph (S, σ) and a map $g\colon \sigma \to S$ such that for each edge $e \in \sigma$ we have that $g(e) \in e$. The map g is called a *pointing* of (S, σ).

Definition (n-complete hypergraph)

Given a set S we denote by $\mathbf{S}_n$ the *n-complete hypergraph* whose vertex set is S and whose edge set is $\bigcup_{k=1}^{n} \binom{S}{k}$.

Hypertournaments

Definition (Hypertournament)

An *n-hypertournament* is a pointed hypergraph $\mathbf{T} := (T, \tau, g)$ where $(T, \tau) = \mathbf{S}_n$ for some set S.

U	0	1	2	01	12	23	34	40	02	13	24	30	41
$g(U)$	0	1	2	1	2	3	4	0	0	1	2	3	4

U	012	123	234	340	401	013	124	230	341	402
$g(U)$	0	1	2	3	4	0	1	2	3	4

RPS$(5, 3)$ example

Hypertournament Magmas

Definition (Hypertournament magma)

Given an n-hypertournament $\mathbf{T} := (T, \tau, g)$ the *hypertournament magma* obtained from $\mathbf{T}$ is the n-magma $\mathbf{A} := (T, f)$ where for $u_1, \ldots, u_n \in T$ we define

$$f(u_1, \ldots, u_n) := g(\{u_1, \ldots, u_n\}).$$

Definition (Hypertournament magma)

A *hypertournament magma* is an n-magma which is conservative and essentially polyadic.

Tournaments

- Tournaments are the $n = 2$ case of a hypertournament.
- Hedrlín and Chvátal introduced the $n = 2$ case of a hypertournament magma in 1965.
- There has been a lot of work on varieties generated by tournament magmas. See for example the survey by Crvenković et al. (1999).

Class Containment Relationships

Proposition

Let $n > 1$. *We have that* $\mathrm{RPS}_n \subsetneq \mathrm{PRPS}_n$, $\mathrm{RPS}_2 \subsetneq \mathrm{Tour}_n$, *and neither of* PRPS_n *and* Tour_n *contains the other. Moreover,* $\mathrm{RPS}_n = \mathrm{PRPS}_n \cap \mathrm{Tour}_n$.

A Generation Result

Theorem

Let $n > 1$. We have that $\mathcal{T}_n = \mathcal{R}_n$. Moreover $\mathcal{T}_n$ is generated by the class of finite regular RPS_n *magmas.*

Proof.

Every finite hypertournament can be embedded in a finite regular balanced hypertournament. $\square$

Counting Regular RPS Magmas

Theorem

Let $m, n \in \mathbb{N}$ with $m \neq 1$ and $n < \varpi(m)$. Given a group $\mathbf{G}$ of order m we have that

$$|\mathrm{RPS}(\mathbf{G}, n)| = \prod_{k=1}^{n} k^{\frac{1}{m}\binom{m}{k}}.$$

Automorphisms

Proposition

Let $\mathbf{A} := \mathbf{G}_n(\lambda)$ *be a regular* RPS *magma. There is a canonical embedding of* $\mathbf{G}$ *into* $\mathbf{Aut}(\mathbf{A})$.

Exceptional Automorphisms

Proposition

For each arity $n \in \mathbb{N}$ *with* $n \neq 1$ *and each group* $\mathbf{G}$ *of composite order* $m \in \mathbb{N}$ *with* $n < \varpi(m)$ *there exists a regular* $\mathrm{RPS}(m, n)$ *magma* $\mathbf{A} := \mathbf{G}_n(\lambda)$ *such that* $|\mathbf{Aut}(\mathbf{A})| > |\mathbf{G}|$.

Proposition

For each arity $n \in \mathbb{N}$ *and each odd prime* p *such that* $1 \neq n \leq p - 2$ *there exists a regular* $\mathrm{RPS}(p, n)$ *magma* $\mathbf{A} := (\mathbb{Z}_p)_n(\lambda)$ *such that* $|\mathbf{Aut}(\mathbf{A})| > |\mathbf{G}|$.

No Exceptional Automorphisms

Proposition

For each odd prime p *and any* $\lambda \in \mathrm{Sgn}_{p-1}(\mathbb{Z}_p)$ *we have that* $\mathbf{Aut}((\mathbb{Z}_p)_{p-1}(\lambda)) \cong \mathbb{Z}_p$.

Corollary

Given an odd prime p *the number of isomorphism classes of magmas of the form* $(\mathbb{Z}_p)_{p-1}(\lambda)$ *is*

$$\prod_{k=1}^{p-1} k^{\frac{1}{p}\binom{p}{k}-1}.$$

For $p = 3$ we have 1, for $p = 5$ we have 6, and for $p = 7$ we have 2073600.

Congruences

Theorem

Let $\theta \in \mathrm{Con}(\mathbf{A})$ *for a regular* $\mathrm{RPS}(m, n)$ *magma* $\mathbf{A} := \mathbf{G}_n(\lambda)$*. Given any* $a \in A$ *we have that* $a/\theta = aH$ *for some subgroup* $\mathbf{H} \leq \mathbf{G}$*.*

λ-convex subgroups

Definition (λ-convex subgroup)

Given a group $\mathbf{G}$, an n-sign function $\lambda \in \mathrm{Sgn}_n(\mathbf{G})$, and a subgroup $\mathbf{H} \leq \mathbf{G}$ we say that $\mathbf{H}$ is λ-*convex* when there exists some $a \in G$ such that $a/\theta = aH$ for some $\theta \in \mathrm{Con}(\mathbf{G}_n(\lambda))$.

λ-convex subgroups

Proposition

Let $\mathbf{G}$ *be a finite group of order* m *and let* $n < \varpi(m)$. *Take* $\lambda \in \mathrm{Sgn}_n(\mathbf{G})$ *and* $\mathbf{H} \leq \mathbf{G}$. *The following are equivalent:*

1. *The subgroup* $\mathbf{H}$ *is* λ*-convex.*
2. *There exists a congruence* $\psi \in \mathrm{Con}(\mathbf{G}_n(\lambda))$ *such that* $e/\psi = H$.
3. *Given* $1 \leq k \leq n-1$ *and* $b_1, \ldots, b_k \notin H$ *either* $e \to \{b_1h_1, \ldots, b_kh_k\}$ *for every choice of* $h_1, \ldots, h_k \in H$ *or* $\{b_1h_1, \ldots, b_kh_k\} \to e$ *for every choice of* $h_1, \ldots, h_k \in H$.

λ-convex subgroups

Theorem

Suppose that $\mathbf{H}, \mathbf{K} \leq \mathbf{G}$ *are both* λ*-convex. We have that* $\mathbf{H} \leq \mathbf{K}$ *or* $\mathbf{K} \leq \mathbf{H}$.

λ-cost Poset

Definition (λ-coset poset)

Given $\lambda \in \mathsf{Sgn}_n(\mathbf{G})$ set

$$P_\lambda := \{\, aH \mid a \in G \text{ and } \mathbf{H} \text{ is } \lambda\text{-convex} \,\}$$

and define the *λ-coset poset* to be $\mathbf{P}_\lambda := (P_\lambda, \subset)$.

Lattices of Maximal Antichains

Dilworth showed that the maximal antichains of a finite poset form a distributive lattice. Freese (1974) gives a succinct treatment of this. Given a finite poset $\mathbf{P} := (P, \leq)$ let $\mathbf{L}(\mathbf{P})$ be the lattice whose elements are maximal antichains in $\mathbf{P}$ where if $U, V \in L(\mathbf{P})$ then we say that $U \leq V$ in $\mathbf{L}(\mathbf{P})$ when for every $u \in U$ there exists some $v \in V$ such that $u \leq v$ in $\mathbf{P}$.

Theorem

We have that $\mathbf{Con}(\mathbf{G}_n(\lambda)) \cong \mathbf{L}(\mathbf{P}_\lambda)$.

A Family of Simple Magmas

Theorem

Suppose that $\mathbf{G} = \mathbb{Z}_{p^k}$ *for a prime* p *and* $n < p$. *There exists a* $\lambda \in \mathrm{Sgn}_n(\mathbf{G})$ *for which* $\mathbf{G}_n(\lambda)$ *is simple.*

A Family of Simple Magmas

Proof.

Order the nontrivial subgroups of $\mathbf{G}$ as $\mathbf{H}_1 \leq \cdots \leq \mathbf{H}_k = \mathbf{G}$. For each $1 \leq i \leq k-1$ choose a coset $a + H_i$ of H_i other than H_i itself which lies in H_{i+1}. Choose another element $b \in a + H_i$ with $b \neq a$. Set $\lambda(\{a, -a\}) := a$ and $\lambda(\{b, -b\}) := -b$. We have that $\mathbf{H}_i$ is not λ-convex for $1 \leq i \leq k-1$. It follows that $\mathbf{G}_n(\lambda)$ has no nontrivial proper λ-convex subgroups for this choice of λ so $\mathbf{G}_n(\lambda)$ is simple. □

Thank you.