Math 2001
Discrete Mathematics
Week 2
Functions

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Today's topics

1. Set operations
2. Functions

Set operations

- The *product* of A and B consists of all ordered pairs (x, y) where $x \in A$ and $y \in B$:

$$A \times B = \{ (x, y) \mid x \in A \text{ and } y \in B \}$$

Set operations

- Note that $|A \uplus B| = |A| + |B|$ for all sets A and B while $|A \times B| = |A| \cdot |B|$ for all sets A and B.
- These operations are also associative, but the product is not commutative.
- Note that $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ in this sense.

Functions

- Informally, a function is a rule that assigns inputs to outputs.
- For each possible input x, a function should give us exactly one output, say y.
- What is a "rule" exactly?
- Both $f(x) = 1$ and $g(x) = \cos^2(x) + \sin^2(x)$ map each real number x to the number 1. Are these the same function?

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Functions

- Our answer to this question will be yes, although we need to specify a little more information.
- Given sets A and B, a *function* is a triple $(A, B, f^{\square})$ where $f^{\square}$ is a subset of $A \times B$ such that for each $a \in A$ there exists exactly one $b \in B$ with $(a, b) \in f^{\square}$.

Functions

- We often write $f\colon A \to B$ to mean that $(A, B, f^{\square})$ is a function.
- The set A is called the *domain* of f, the set B is called the *codomain* of f (**not** the range of f), and the set $f^{\square}$ is sometimes called the *graph* of f (like in calculus).
- We write $f(a) = b$ to mean that $(a, b) \in f^{\square}$.

Functions

- If we define $f\colon \mathbb{R} \to \mathbb{R}$ by the rule that $f(x) = 1$ and $g\colon \mathbb{R} \to \mathbb{R}$ by the rule that $g(x) = \cos^2(x) + \sin^2(x)$ then we see that $f^{\square} = g^{\square}$. Since f and g have the same domain, f and g have the same codomain, and $f^{\square} = g^{\square}$, we say that f and g are the same function.
- We might write $f = g$ in this case.

Functions

- Let $h\colon \mathbb{R} \to \mathbb{N}$ be defined by $h(x) = 1$. Is $f = h$?
- According to our idea that a function consists of its domain, codomain, and graph, the answer is no.
- The function f is $(\mathbb{R}, \mathbb{R}, f^{\square})$ and the function h is $(\mathbb{R}, \mathbb{N}, h^{\square})$.
- Even though $f^{\square} = h^{\square}$, $\mathbb{R} \neq \mathbb{N}$, so $f \neq h$.

Functions

- While the is an "obvious" correspondence between f and h, we will benefit from keeping them distinct. (If you code, think about the troubles that implicit type casting can cause if you're not careful.)

Functions

- Some things that you called functions on $\mathbb{R}$ in calculus may not be functions on $\mathbb{R}$ in this sense.
- Consider the rule $f(x) = \frac{1}{x}$. There is no real number corresponding to $f(0) = \frac{1}{0}$, so we cannot have a function $f\colon \mathbb{R} \to \mathbb{R}$ given by this rule.
- There is such a function $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, and this is usually what we mean by $f(x) = \frac{1}{x}$.

Functions

- When our domain is infinite, we have little choice but to give a function by a rule like $f(x) = x^2 + e^x$.
- When our domain is finite, we may just give the output value of the function for each possible input.
- We can present this as a set of pairs, which we often write as a table.