Math 2130
Linear Algebra
Week 8
Matrix multiplication

Charlotte Aten

2025 October 17

Today's topics

1. Matrix multiplication

Matrix multiplication

- Let A, B, and C be matrices of the appropriate sizes so that the indicated operations may be performed. All of the following are true:
- $A(BC) = (AB)C$ (matrix multiplication is associative)
- $A(B + C) = AB + AC$ (matrix multiplication on the left distributes over addition)
- $(A + B)C = AC + BC$ (matrix multiplication on the right distributes over addition)

Matrix multiplication

- We use the same notation for multiplying a matrix by itself as we do for multiplying a scalar by itself.
- That is,

$$A^2 = AA,\ A^3 = AAA,\ A^4 = AAAA,$$

and so forth.

Matrix multiplication

Given that

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

compute A^3.

Matrix multiplication

Given that

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

compute A^3.
Since $A^3 = A^2 A$ and

$$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

we have that

$$A^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Matrix multiplication

Definition

The $n \times n$ matrix

$$I_n = [\delta_{ij}]$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{otherwise} \end{cases}$$

is called the *identity matrix* of size $n \times n$.

- We have that

$$I_1 = \begin{bmatrix} 1 \end{bmatrix},\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \text{and } I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Matrix multiplication

- The identity matrix plays the same role in matrix multiplication that 1 does in the usual multiplication.
- If A is any $m \times n$ matrix then

$$I_m A = A = A I_n.$$

Matrix multiplication

- Note that if we identify 1 with $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $i = \sqrt{-1}$ with $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ then we can think of complex numbers as matrices.
- That is, $a + bi$ can be identified with $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and addition and multiplication of complex numbers is compatible with this identification.

Matrix multiplication

In the usual multiplication we have that 1 has only two square roots, which are 1 and -1.
Find all matrices A of the form

$$A = \begin{bmatrix} a & 1 \\ -1 & b \end{bmatrix}$$

such that $A^2 = I_2$.

Matrix multiplication

In the usual multiplication we have that 1 has only two square roots, which are 1 and -1.
Find all matrices A of the form

$$A = \begin{bmatrix} a & 1 \\ -1 & b \end{bmatrix}$$

such that $A^2 = I_2$.
We have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 = A^2 = \begin{bmatrix} a^2 - 1 & a + b \\ -a - b & -1 + b^2 \end{bmatrix}$$

so $b = -a$ and $a = \pm\sqrt{2}$.

Matrix multiplication

In the usual multiplication we have that 1 has only two square roots, which are 1 and -1.
Find all matrices A of the form

$$A = \begin{bmatrix} a & 1 \\ -1 & b \end{bmatrix}$$

such that $A^2 = I_2$.
We find that

$$\begin{bmatrix} \sqrt{2} & 1 \\ -1 & -\sqrt{2} \end{bmatrix}^2 = I_2 = \begin{bmatrix} -\sqrt{2} & 1 \\ -1 & \sqrt{2} \end{bmatrix}^2.$$

Matrix multiplication

- We see that the Fundamental Theorem of Algebra is not true for matrix algebra, so it is possible that a given polynomial equation like $x^2 - 1 = 0$ might have more solutions than its degree when we interpret it as a matrix equation $A^2 - I = 0$.

Matrix multiplication

- It is not the case that any finite power of a matrix needs to be the identity matrix.
- For example, consider $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.