Math 2130
Linear Algebra
Week 8
Matrix multiplication

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Today's topics

1. Matrix multiplication

Matrix multiplication

- Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (-y, x)$ and let $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $g(x, y) = (2x, y)$.
- Using the standard basis $B = \{(1, 0), (0, 1)\}$, we can represent f as

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

  and we can represent g as

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Matrix multiplication

- Note that $g \circ f \colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$(g \circ f)(x, y) = g(f(x, y)) = g(-y, x) = (-2y, x),$$

which has matrix representation

$$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}.$$

Matrix multiplication

Definition

Given an $m \times n$ matrix A and an $n \times k$ matrix B where a_i is the i^{th} row of A and b_j is the j^{th} column of B the *product* of A and B is the $m \times k$ matrix

$$AB = [a_i \cdot b_j].$$

- For example, if $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 4 & 7 \end{bmatrix}$ then

$$AB = \begin{bmatrix} 14 & 26 \end{bmatrix}.$$

Matrix multiplication

- We have that $(g \circ f)(x) = (-2y, x)$ and $(f \circ g)(x, y) = (-y, 2x)$.
- Similarly, we have that

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$$

  while

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}.$$

- Typically, we don't have that $AB = BA$ for matrices.

Matrix multiplication

- When $A = [a_{ij}]$ is an $m \times n$ matrix, $B = [b_{ij}]$ is an $n \times k$ matrix, and $AB = C = [c_{ij}]$ we have that

$$c_{ij} = \sum_{\ell=1}^{n} a_{i\ell} b_{\ell j}.$$

- This is called the *index form* of the matrix product.
- Observe that

$$\sum_{\ell=1}^{n} a_{i\ell} b_{\ell j} = a_i \cdot b_j,$$

so this agrees with the formula we already gave.