Math 2130
Linear Algebra
Week 8
Matrices and isomorphisms

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Today's topics

1. Dimension and isomorphism
2. Matrix multiplication

Dimension and isomorphism

The function $f\colon \mathbb{R}^2 \to \mathcal{P}_1$ given by

$$f(a,b) = (3a+3b)x + (3a+b)$$

is an isomorphism. We have that

$$\{f(1,0), f(0,1)\} = \{3x+3, 3x+1\}$$

is a basis for $\mathcal{P}_1$.

Dimension and isomorphism

Definition (Dimension)

The *dimension* of a vector space V is the number of vectors in a basis for V.

Dimension and isomorphism

Theorem

The vector spaces V and W are isomorphic if and only if $\dim(V) = \dim(W)$.

Matrix multiplication

- Note that when $f\colon V \to W$ is an isomorphism and $B = \{b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_n\}$ are bases for V and W, respectively, we only need to know how to write $f(b_i)$ as a linear combination of the c_j in order to know what f does to all vectors in V.
- This means that we can represent f as a matrix.

Matrix multiplication

- Given two isomorphisms $f\colon U \to V$ and $g\colon V \to W$, let's see what the matrix for $g \circ f$ looks like.

Matrix multiplication

- Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x,y) = (-y,x)$ and let $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $g(x,y) = (2x,y)$.
- Using the standard basis $B = \{(1,0),(0,1)\}$, we can represent f as

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and we can represent g as

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Matrix multiplication

- Note that $g \circ f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$(g \circ f)(x, y) = g(f(x, y)) = g(-y, x) = (-2y, x),$$

which has matrix representation

$$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}.$$