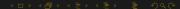
Math 2130 Linear Algebra Week 8 Matrices and isomorphisms

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Today's topics

- Dimension and isomorphism
- Matrix multiplication

Dimension and isomorphism

The function $f: \mathbb{R}^2 \to \mathcal{P}_1$ given by

$$f(a,b) = (3a+3b)x + (3a+b)$$

is an isomorphism. We have that

$${f(1,0), f(0,1)} = {3x + 3, 3x + 1}$$

is a basis for \mathcal{P}_1 .

Dimension and isomorphism

Definition (Dimension)

The dimension of a vector space V is the number of vectors in a basis for V.

Dimension and isomorphism

Theorem

The vector spaces V and W are isomorphic if and only if $\dim(V) = \dim(W)$.

- Note that when $f\colon V\to W$ is an isomorphism and $B=\{b_1,\ldots,b_n\}$ and $C=\{c_1,\ldots,c_n\}$ are bases for V and W, respectively, we only need to know how to write $f(b_i)$ as a linear combination of the c_j in order to know what f does to all vectors in V.
- \blacksquare This means that we can represent f as a matrix.

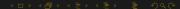
■ Given two isomorphisms $f \colon U \to V$ and $g \colon V \to W$, let's see what the matrix for $g \circ f$ looks like.

- Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be given by f(x,y) = (-y,x) and let $g: \mathbb{R}^2 \to \mathbb{R}^2$ be given by g(x,y) = (2x,y).
- Using the standard basis $B = \{(1,0),(0,1)\}$, we can represent f as

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and we can represent q as

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$



■ Note that $g \circ f : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$(g \circ f)(x, y) = g(f(x, y)) = g(-y, x) = (-2y, x),$$

which has matrix representation

$$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}.$$