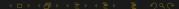
Math 2130 Linear Algebra Week 7 Isomorphisms

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Today's topics

- Dimension
- 2 Bijections
- 3 Isomorphisms

Definition (Row space)

The *row space* of an $m \times n$ matrix A is the subspace of \mathbb{R}^n spanned by the rows of A. We denote the row space of A by $\mathrm{Row}(A)$.

- The rank of a matrix A is $\dim(\text{Row}(A))$.
- lacktriangle This is the same as the number of leading 1s in any echelon form of A.

Definition (Column space)

The *column space* of an $m \times n$ matrix A is the subspace of \mathbb{R}^m spanned by the columns of A. We denote the column space of A by $\operatorname{Col}(A)$.

- The rank of a matrix A is equal to $\dim(\operatorname{Col}(A))$.
- Note that $\operatorname{Row}(A)$ and $\operatorname{Col}(A)$ are subspaces of \mathbb{R}^n and \mathbb{R}^m , respectively, so they contain vectors with different numbers of entries when $m \neq n$.

- When A is the coefficient matrix of a system with n unknowns, the solutions to the corresponding homogeneous system (i.e. all constants set to 0) form a vector space whose dimension is $n \operatorname{Rank}(A)$.
- When A is an $n \times n$ matrix, this means that $\operatorname{Rank}(A) = n$ when there is exactly one solution to the corresponding homogeneous system.

- A set of n vectors in \mathbb{R}^n is a basis precisely when the $n \times n$ matrix consisting of those vectors (as either its rows or columns) has rank n.
- This means a set of n vectors in \mathbb{R}^n is a basis exactly when the reduced echelon form of the matrix consisting of those vectors is the $n \times n$ identity matrix I_n .

Bijections¹

Definition

We say that a function $f:A\to B$ is injective (or one-to-one) when $f(a_1)=f(a_2)$ implies that $a_1=a_2.$

Bijections

Definition

We say that a function $f: A \to B$ is *surjective* (or *onto*) when for each $b \in B$ there is some $a \in A$ for which f(a) = b.

Bijections

Definition

We say that a function $f: A \to B$ is a bijection (or a one-to-one correspondence) when f is both injective and surjective.

Isomorphisms

Definition

Given vector spaces V and W we say that a function $f \colon V \to W$ is an isomorphism between V and W when

- f is a bijection,
- **2** for all $v_1, v_2 \in V$ we have $f(v_1 + v_2) = f(v_1) + f(v_2)$, and
- for all $s \in \mathbb{R}$ and all $v \in V$ we have f(sv) = sf(v).