Math 2130
Linear Algebra
Week 7
Isomorphisms

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Today's topics

1. Dimension
2. Bijections
3. Isomorphisms

Dimension

Definition (Row space)

The *row space* of an $m \times n$ matrix A is the subspace of $\mathbb{R}^n$ spanned by the rows of A. We denote the row space of A by $\mathrm{Row}(A)$.

Dimension

- The *rank* of a matrix A is $\dim(\mathrm{Row}(A))$.
- This is the same as the number of leading 1s in any echelon form of A.

Dimension

Definition (Column space)

The *column space* of an $m \times n$ matrix A is the subspace of $\mathbb{R}^m$ spanned by the columns of A. We denote the column space of A by $\mathrm{Col}(A)$.

Dimension

- The *rank* of a matrix A is equal to $\dim(\mathrm{Col}(A))$.
- Note that $\mathrm{Row}(A)$ and $\mathrm{Col}(A)$ are subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, so they contain vectors with different numbers of entries when $m \neq n$.

Dimension

- When A is the coefficient matrix of a system with n unknowns, the solutions to the corresponding homogeneous system (i.e. all constants set to 0) form a vector space whose dimension is $n - \mathrm{Rank}(A)$.
- When A is an $n \times n$ matrix, this means that $\mathrm{Rank}(A) = n$ when there is exactly one solution to the corresponding homogeneous system.

Dimension

- A set of n vectors in $\mathbb{R}^n$ is a basis precisely when the $n \times n$ matrix consisting of those vectors (as either its rows or columns) has rank n.
- This means a set of n vectors in $\mathbb{R}^n$ is a basis exactly when the reduced echelon form of the matrix consisting of those vectors is the $n \times n$ *identity matrix* I_n.

Bijections

Definition

We say that a function $f\colon A \to B$ is *injective* (or *one-to-one*) when $f(a_1) = f(a_2)$ implies that $a_1 = a_2$.

Bijections

Definition

We say that a function $f\colon A \to B$ is *surjective* (or *onto*) when for each $b \in B$ there is some $a \in A$ for which $f(a) = b$.

Bijections

Definition

We say that a function $f\colon A \to B$ is a *bijection* (or a *one-to-one correspondence*) when f is both injective and surjective.

Isomorphisms

Definition

Given vector spaces V and W we say that a function $f\colon V \to W$ is an *isomorphism* between V and W when

1. f is a bijection,
2. for all $v_1, v_2 \in V$ we have $f(v_1 + v_2) = f(v_1) + f(v_2)$, and
3. for all $s \in \mathbb{R}$ and all $v \in V$ we have $f(sv) = sf(v)$.