Math 2130
Linear Algebra
Week 6
Bases

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Today's topics

1. Bases

Bases

Definition

A set of vectors $\{v_1, v_2, \ldots, v_k\}$ in a vector space V is a *basis* for V when

1. $\mathrm{Span}(\{v_1, v_2, \ldots, v_k\}) = V$ and
2. $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

Bases

- A set of vectors S is a spanning set for a vector space V when each vector in V be be written as a linear combination of the vectors from S in **at least** one way.
- A set of vectors S is linearly independent in a vector space V when each vector in V can be written as a linear combination of the vectors from S in **at most** one way.
- A set of vectors S is a basis for V when each vector in V can be written as a linear combination of the vectors in S in **exactly** one way.

Bases

Definition

The *standard basis* for $\mathbb{R}^n$ is $\{e_1, e_2, \ldots, e_n\}$ where $(e_i)_j = \delta_{ij}$.

- The standard basis for $\mathbb{R}^2$ is $\{(1,0),(0,1)\}$.
- The standard basis for $\mathbb{R}^3$ is $\{(1,0,0),(0,1,0),(0,0,1)\}$.

Bases

- There are other bases for $\mathbb{R}^n$ besides the standard one.
- In $\mathbb{R}^2$ we have that $\{(1,0),(1,1)\}$ is a basis.
- Similarly, in $\mathbb{R}^3$ we have that $\{(0,0,1),(0,1,1),(1,1,1)\}$ is a basis.

Bases

Definition

The *standard basis* for $\mathrm{Mat}_{m\times n}$ is

$$\{\, E_{uv} \mid 1 \leq u \leq m \text{ and } 1 \leq v \leq n \,\}$$

where $E_{uv} = [e_{ij}]$ is defined to have $e_{ij} = 1$ when $u = i$ and $v = j$ and $e_{ij} = 0$ otherwise.

- For example, the standard basis for $\mathrm{Mat}_{2\times 3}$ is

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Bases

Definition

The *standard basis* of P_n is

$$\{1, x, x^2, \ldots, x^n\}.$$

Bases

Show that

$$\{x^2 + x + 1, x^2 + x + 2, x^2 + 2x + 3\}$$

is a basis for P_2.

Bases

Show that

$$\{x^2 + x + 1, x^2 + x + 2, x^2 + 2x + 3\}$$

is a basis for P_2.
One can check that any quadratic $a_2x^2 + a_1x + a_0$ can be written as

$$c_1(x^2 + x + 1) + c_2(x^2 + x + 2) + c_3(x^2 + 2x + 3),$$

equating coefficients, and solving the resulting linear system for c.
A similar calculation shows that the only linear combination of the given set of polynomials which yields 0 is the trivial one.

Dimension

Definition

The *dimension* of a vector space V is the size of any basis for V.

Dimension

- The dimension of $\mathbb{R}^n$ is n.
- The dimension of $\mathrm{Mat}_{m\times n}$ is mn.
- The dimension of $\mathcal{P}_n$ is $n+1$.