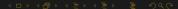
Math 2130 Linear Algebra Week 6 Bases

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Today's topics

1 Bases

Definition

A set of vectors $\{v_1, v_2, \dots, v_k\}$ in a vector space V is a *basis* for V when

- A set of vectors *S* is a spanning set for a vector space *V* when each vector in *V* be be written as a linear combination of the vectors from *S* in **at least** one way.
- A set of vectors S is linearly independent in a vector space V when each vector in V can be written as a linear combination of the vectors from S in **at most** one way.
- A set of vectors S is a basis for V when each vector in V can be written as a linear combination of the vectors in S in **exactly** one way.

Definition

The standard basis for \mathbb{R}^n is $\{e_1, e_2, \dots, e_n\}$ where $(e_i)_j = \delta_{ij}$.

- The standard basis for \mathbb{R}^2 is $\{(1,0),(0,1)\}$.
- The standard basis for \mathbb{R}^3 is $\{(1,0,0),(0,1,0),(0,0,1)\}.$

- There are other bases for \mathbb{R}^n besides the standard one.
- In \mathbb{R}^2 we have that $\{(1,0),(1,1)\}$ is a basis.
- Similarly, in \mathbb{R}^3 we have that $\{(0,0,1),(0,1,1),(1,1,1)\}$ is a basis.

Definition

The standard basis for $Mat_{m \times n}$ is

$$\{E_{uv} \mid 1 \leq u \leq m \text{ and } 1 \leq v \leq n\}$$

where $E_{uv}=\left[e_{ij}\right]$ is defined to have $e_{ij}=1$ when u=i and v=j and $e_{ij}=0$ otherwise.

■ For example, the standard basis for $Mat_{2\times3}$ is

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Definition

The standard basis of P_n is

$$\left\{1, x, x^2, \dots, x^n\right\}.$$

Show that

$${x^2 + x + 1, x^2 + x + 2, x^2 + 2x + 3}$$

is a basis for P_2 .

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is a basis for P_2 .

One can check that any quadratic $a_2x^2+a_1x+a_0$ can be written as

$$c_1(x^2 + x + 1) + c_2(x^2 + x + 2) + c_3(x^2 + 2x + 3),$$

equating coefficients, and solving the resulting linear system for c. A similar calculation shows that the only linear combination of the given set of polynomials which yields 0 is the trivial one.

Dimension

Definition

The dimension of a vector space V is the size of any basis for V.

Dimension

- The dimension of \mathbb{R}^n is n.
- The dimension of $Mat_{m \times n}$ is mn.
- The dimension of \mathcal{P}_n is n+1.