Math 2130
Linear Algebra
Week 5
Linear independence

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Today's topics

1. Linear independence
2. Other vector spaces beside $\mathbb{R}^n$

Linear independence

Definition

A finite nonempty set of vectors $\{v_1, v_2, \ldots, v_k\}$ in a vector space V is said to be *linearly dependent* when there exist scalars $c_1, c_2, \ldots, c_k$, at least one of which is nonzero, such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0.$$

Definition

A finite nonempty set of vectors $\{v_1, v_2, \ldots, v_k\}$ in a vector space V is said to be *linearly independent* when it is not linearly dependent.

Linear independence

- In $\mathbb{R}^2$ the spanning sets $\{(1,0),(0,1)\}$ and $\{(1,0),(1,1)\}$ are linearly independent. The spanning set $\{(1,0),(0,1),(1,2)\}$ is linearly dependent.
- The spanning set $\{(1,1,1),(1,1,2),(1,2,3)\}$ for $\mathbb{R}^3$ is also linearly independent.
- Even sets which do not span can be linearly independent. For example, $\{(1,1,1),(1,1,2)\}$ is linearly independent in $\mathbb{R}^3$ but does not span.
- Note that any set consisting of a single nonzero vector $\{v\}$ is linearly independent.
- On the other hand, any set containing the zero vector is linearly dependent.

Linear independence

Determine whether $\{(1,4,7),(2,5,8),(3,6,9)\}$ is linearly independent in $\mathbb{R}^3$.

Linear independence

Determine whether $\{(1,4,7),(2,5,8),(3,6,9)\}$ is linearly independent in $\mathbb{R}^3$.
If we have

$$c_1(1,4,7)+c_2(2,5,8)+c_3(3,6,9)=(0,0,0)$$

then we must have a homogeneous system with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right].$$

Linear independence

Determine whether $\{(1,4,7),(2,5,8),(3,6,9)\}$ is linearly independent in $\mathbb{R}^3$.
This matrix has row-echelon form

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so we can see that there is a free variable and hence infinitely many solutions. Thus, the set $\{(1,4,7),(2,5,8),(3,6,9)\}$ is linearly dependent.

Other vector spaces beside $\mathbb{R}^n$

- We can talk about spanning sets and linear independence for vector spaces other than $\mathbb{R}^n$.
- For example, $\{2x+1, 5, x^2+x+1\}$ is a spanning set for $\mathcal{P}_2$.