Math 2130
Linear Algebra
Week 5
Spanning sets and linear independence

Charlotte Aten

2025 September 24

Today's topics

1. Spanning sets
2. Linear independence

Spanning sets

Definition

If every vector in a vector space V can be written as a linear combination of $\{v_1, v_2, \ldots, v_k\}$ we say that V is *spanned* (or *generated*) by $\{v_1, v_2, \ldots, v_k\}$. The set of vectors $\{v_1, v_2, \ldots, v_k\}$ is called a *spanning set* for V. We also say that $\{v_1, v_2, \ldots, v_k\}$ *spans* V and write $\mathrm{Span}(\{v_1, v_2, \ldots, v_k\}) = V$ in this situation.

- More generally, we define $\mathrm{Span}(\{v_1, v_2, \ldots, v_k\})$

$$\{\, c_1 v_1 + c_2 v_2 + \cdots + c_k v_k \mid c_1, c_2, \ldots, c_k \in \mathbb{R} \,\}.$$

Spanning sets

Theorem

Given a set of vectors $\{v_1, v_2, \ldots, v_k\}$ *where* v_k *is a linear combination of* $\{v_1, v_2, \ldots, v_{k-1}\}$ *we have that*

$$\mathrm{Span}(\{v_1, v_2, \ldots, v_k\}) = \mathrm{Span}(\{v_1, v_2, \ldots, v_{k-1}\}).$$

Spanning sets

- Both $\{(1,0),(0,1)\}$ and $\{(1,0),(1,1)\}$ are spanning sets for $\mathbb{R}^2$. By similar reasoning, or by the previous theorem, so is $\{(1,0),(0,1),(1,2)\}$.

Linear independence

Definition

A finite nonempty set of vectors $\{v_1, v_2, \ldots, v_k\}$ in a vector space V is said to be *linearly dependent* when there exist scalars $c_1, c_2, \ldots, c_k$, at least one of which is nonzero, such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0.$$

Definition

A finite nonempty set of vectors $\{v_1, v_2, \ldots, v_k\}$ in a vector space V is said to be *linearly independent* when it is not linearly dependent.

Linear independence

Theorem

A finite nonempty set of vectors $\{v_1, v_2, \ldots, v_k\}$ *in a vector space* V *is linearly independent when the only choice of scalars* $c_1, c_2, \ldots, c_k$ *so that*

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

is $c_1 = c_2 = \cdots = c_k = 0.$

Linear independence

- In $\mathbb{R}^2$ the spanning sets $\{(1,0),(0,1)\}$ and $\{(1,0),(1,1)\}$ are linearly independent. The spanning set $\{(1,0),(0,1),(1,2)\}$ is linearly dependent.