Math 2130
Linear Algebra
Week 5
Subspaces

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Today's topics

1. Subspaces
2. Spanning sets

Subspaces

Definition

Given a vector space V and a nonempty subset S of V we say that S is a *subspace* of V when S is a vector space under the same addition and scalar multiplication used in V.

Subspaces

- Fortunately, we don't have to check all ten axioms discussed previously in order to see if S is a really a subspace of V.

Theorem (Subspace test)

A nonempty subset S of a vector space V is a subspace of V if and only if S is closed under the addition and scalar multiplication operations of V.

Subspaces

Use the Subspace Test to show that

$$\{ (x, y) \in \mathbb{R}^2 \mid (x, y) \cdot (3, -1) = 0 \}$$

is a subspace of $\mathbb{R}^2$.

Subspaces

Use the Subspace Test to show that

$$\{\, (x, y) \in \mathbb{R}^2 \mid (x, y) \cdot (3, -1) = 1 \,\}$$

is not a subspace of $\mathbb{R}^2$. (Note that it is a subset of $\mathbb{R}^2$, but not a subspace.)

Spanning sets

- Consider the linear system

$$\begin{aligned} x_1 + 3x_3 + x_4 &= 0 \\ x_2 - x_3 - x_4 &= 0. \end{aligned}$$

- The augmented matrix for this system is

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 \end{array}\right],$$

which is already in reduced row-echelon form.
- We can therefore take free variables $x_3 = u$ and $x_4 = v$.
- We have that $x_1 = -3u - v$ and $x_2 = u + v$.

Spanning sets

- We find that solutions for the linear system

$$\begin{aligned} x_1 + 3x_3 + x_4 &= 0 \\ x_2 - x_3 - x_4 &= 0 \end{aligned}$$

  are of the form

$$(-3u - v, u + v, u, v)$$

  where $u, v \in \mathbb{R}$.
- The solution set

$$S = \{ (-3u - v, u + v, u, v) \in \mathbb{R}^4 \mid u, v \in \mathbb{R} \}$$

  is a subspace of $\mathbb{R}^4$.

Spanning sets

- Note that a member of

  $$S = \{\, (-3u - v, u + v, u, v) \in \mathbb{R}^4 \mid u, v \in \mathbb{R} \,\}$$

  can be written as

  $$(-3u - v, u + v, u, v) = u(-3, 1, 1, 0) + v(-1, 1, 0, 1)$$

  for some $u, v \in \mathbb{R}$.
- In this sense, every solution to the linear system is made from the two basic solutions $x = (-3, 1, 1, 0)$ and $x = (-1, 1, 0, 1)$.

Spanning sets

- Note that if we have a set of vectors $\{v_1, v_2, \ldots, v_k\}$ in a vector space V then the most general way they can be combined is

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

  where $c_1, c_2, \ldots, c_k$ are scalars.
- The previous discussion was then about the fact that every solution to the given system was a linear combination of $(-3, 1, 1, 0)$ and $(-1, 1, 0, 1)$.

Spanning sets

Definition

If every vector in a vector space V can be written as a linear combination of $\{v_1, v_2, \ldots, v_k\}$ we say that V is *spanned* (or *generated*) by $\{v_1, v_2, \ldots, v_k\}$. The set of vectors $\{v_1, v_2, \ldots, v_k\}$ is called a *spanning set* for V. We also say that $\{v_1, v_2, \ldots, v_k\}$ *spans* V and write $\mathrm{Span}(\{v_1, v_2, \ldots, v_k\}) = V$ in this situation.

- More generally, we define $\mathrm{Span}(\{v_1, v_2, \ldots, v_k\})$

$$\{\, c_1 v_1 + c_2 v_2 + \cdots + c_k v_k \mid c_1, c_2, \ldots, c_k \in \mathbb{R} \,\}.$$