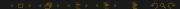
Math 2130 Linear Algebra Week 5 Subspaces

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Today's topics

- Subspaces
- Spanning sets

Definition

Given a vector space V and a nonempty subset S of V we say that S is a *subspace* of V when S is a vector space under the same addition and scalar multiplication used in V.

■ Fortunately, we don't have to check all ten axioms discussed previously in order to see if S is a really a subspace of V.

Theorem (Subspace test)

A nonempty subset S of a vector space V is a subspace of V if and only if S is closed under the addition and scalar multiplication operations of V.

Use the Subspace Test to show that

$$\{(x,y) \in \mathbb{R}^2 \mid (x,y) \cdot (3,-1) = 0\}$$

is a subspace of \mathbb{R}^2 .

Use the Subspace Test to show that

$$\{(x,y) \in \mathbb{R}^2 \mid (x,y) \cdot (3,-1) = 1\}$$

is not a subspace of \mathbb{R}^2 . (Note that it is a subset of \mathbb{R}^2 , but not a subspace.)

Consider the linear system

$$x_1 + 3x_3 + x_4 = 0$$
$$x_2 - x_3 - x_4 = 0.$$

■ The augmented matrix for this system is

$$\begin{bmatrix} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 \end{bmatrix},$$

which is already in reduced row-echelon form.

- We can therefore take free variables $x_3 = u$ and $x_4 = v$.
- We have that $x_1 = -3u v$ and $x_2 = u + v$.



We find that solutions for the linear system

$$x_1 + 3x_3 + x_4 = 0$$
$$x_2 - x_3 - x_4 = 0$$

are of the form

$$(-3u - v, u + v, u, v)$$

where $u, v \in \mathbb{R}$.

The solution set

$$S = \{ (-3u - v, u + v, u, v) \in \mathbb{R}^4 \mid u, v \in \mathbb{R} \}$$

is a subspace of \mathbb{R}^4 .



Note that a member of

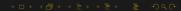
$$S = \{ (-3u - v, u + v, u, v) \in \mathbb{R}^4 \mid u, v \in \mathbb{R} \}$$

can be written as

$$(-3u - v, u + v, u, v) = u(-3, 1, 1, 0) + v(-1, 1, 0, 1)$$

for some $u, v \in \mathbb{R}$.

■ In this sense, every solution to the linear system is made from the two basic solutions x = (-3, 1, 1, 0) and x = (-1, 1, 0, 1).



Note that if we have a set of vectors $\{v_1, v_2, \dots, v_k\}$ in a vector space V then the most general way they can be combined is

$$c_1v_1+c_2v_2+\cdots+c_kv_k$$

where c_1, c_2, \ldots, c_k are scalars.

■ The previous discussion was then about the fact that every solution to the given system was a linear combination of (-3,1,1,0) and (-1,1,0,1).

Definition

If every vector in a vector space V can be written as a linear combination of $\{v_1,v_2,\ldots,v_k\}$ we say that V is spanned (or generated) by $\{v_1,v_2,\ldots,v_k\}$. The set of vectors $\{v_1,v_2,\ldots,v_k\}$ is called a spanning set for V. We also say that $\{v_1,v_2,\ldots,v_k\}$ spans V and write $\mathrm{Span}(\{v_1,v_2,\ldots,v_k\})=V$ in this situation.

lacksquare More generally, we define $\mathrm{Span}(\{v_1,v_2,\ldots,v_k\})$

$$\{c_1v_1+c_2v_2+\cdots+c_kv_k \mid c_1,c_2,\ldots,c_k \in \mathbb{R}\}.$$

