Math 2130
Linear Algebra
Week 4
Vector spaces

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2025 September 19

Today's topics

1. Vector spaces
2. Subspaces

Vector spaces

Definition (Vector space)

Let V be a set (whose members are called *vectors*) on which addition and scalar multiplication are defined. We say that V is a *vector space* when

- (closure under addition) for each pair of vectors u and v in V we have that $u + v \in V$,
- (closure under scalar multiplication) for each vector $v \in V$ and each scalar $s \in \mathbb{R}$ we have that $sv \in V$,
- (commutativity of addition) for all $u, v \in V$ we have that $u + v = v + u$,

Vector spaces

Definition (Vector space)

- (associativity of addition) for all $u, v, w \in V$ we have that $u + (v + w) = (u + v) + w$,
- (zero vector) there is a vector $0 \in V$ such that for any $v \in V$ we have $v + 0 = v$,
- (additive inverse) for each $v \in V$ there is a vector $-v \in V$ such that $v + (-v) = 0$,
- (unit property) for each $v \in V$ we have $1v = v$,
- (associativity of scalar multiplication) for all $v \in V$ and all $r, s \in \mathbb{R}$ we have $(rs)v = r(sv)$,

Vector spaces

Definition (Vector space)

- (distributivity of scalar multiplication over vector addition) for all $u, v \in V$ and all $r \in \mathbb{R}$ we have that $r(u+v) = ru + rv$, and
- (distributivity of scalar multiplication over scalar addition) for all $v \in V$ and all $r, s \in \mathbb{R}$ we have that $(r+s)v = rv + sv$.

Vector spaces

Show that

$$\{\, (x, y) \in \mathbb{R}^2 \;\middle|\; (x, y) \cdot (3, -1) = 1 \,\}$$

is not a vector space under the usual vector operations.

Vector spaces

Show that

$$\left\{ \begin{bmatrix} a & a+b \\ b & b \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$$

is closed under scalar multiplication.

Vector spaces

Show that scalar multiplication distributes over addition for real-valued functions $f\colon \mathbb{R} \to \mathbb{R}$.

Vector spaces

Consider the set of vectors $\mathbb{R}^2$ where we define addition by

$$(x_1, x_2) \oplus (y_1, y_2) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

and scalar multiplication is defined as usual. Show that this is not a vector space.

Subspaces

Definition

Given a vector space V and a nonempty subset S of V we say that S is a *subspace* of V when S is a vector space under the same addition and scalar multiplication used in V.

Subspaces

- Fortunately, we don't have to check all ten axioms discussed previously in order to see if S is a really a subspace of V.

Theorem (Subspace test)

A nonempty subset S of a vector space V is a subspace of V if and only if S is closed under the addition and scalar multiplication operations of V.