Math 2130
Linear Algebra
Week 4
Vector spaces

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Today's topics

1. Vector spaces

Vector spaces

- We have seen many different collections of objects, such as vectors in $\mathbb{R}^n$ and various sized matrices, which can be added as well as multiplied by scalars.
- At this point it may seem reasonable that the arithmetic of matrices should be thought of as some kind of generalization of the arithmetic of real numbers themselves.
- The concept of a vector space encompasses this idea, as well as several others we will use throughout the rest of the course.

Examples of vector spaces

- The vectors in $\mathbb{R}^n$ where $(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1 +, \dots, x_n + y_n)$ and $s(x_1, \dots, x_n) = (sx_1, \dots, sx_n)$.
- The numbers in $\mathbb{R}$ with the usual addition and multiplication. Note that $\mathbb{R} = \mathbb{R}^1$.
- The matrices $\mathrm{Mat}_{m \times n}$ of size $m \times n$ where addition and scalar multiplication are done componentwise.

Examples of vector spaces

- The set of all functions $f\colon \mathbb{R} \to \mathbb{R}$ where $(f+g)(x) = f(x) + g(x)$ and $(sf)(x) = s(f(x))$.
- The set of all functions $f\colon \mathbb{R}^n \to \mathbb{R}$ where $(f+g)(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) + g(x_1, \ldots, x_n)$ and $(sf)(x_1, \ldots, x_n) = s(f(x_1, \ldots, x_n))$.

Examples of vector spaces

- The set of all quadratic polynomials $\mathcal{P}_2 = \{ ax^2 + bx + c \mid a, b, c \in \mathbb{R} \}$ with $(a_1x^2 + b_1x + c_1) + (a_2x^2 + b_2x + c_2) = (a_1 + a_2)x^2 + (b_1 + b_2)x + (c_1 + c_2)$ and $s(ax^2 + bx + c) = (sa)x^2 + (sb)x + (sc)$.
- The set of all degree at most n polynomials, with addition and scalar multiplication defined similarly. This vector space is denoted by $\mathcal{P}_n$.

Vector spaces

Definition (Vector space)

Let V be a set (whose members are called *vectors*) on which addition and scalar multiplication are defined. We say that V is a *vector space* when

- (closure under addition) for each pair of vectors u and v in V we have that $u + v \in V$,
- (closure under scalar multiplication) for each vector $v \in V$ and each scalar $s \in \mathbb{R}$ we have that $sv \in V$,
- (commutativity of addition) for all $u, v \in V$ we have that $u + v = v + u$,

Vector spaces

Definition (Vector space)

- (associativity of addition) for all $u, v, w \in V$ we have that $u + (v + w) = (u + v) + w$,
- (zero vector) there is a vector $0 \in V$ such that for any $v \in V$ we have $v + 0 = v$,
- (additive inverse) for each $v \in V$ there is a vector $-v \in V$ such that $v + (-v) = 0$,
- (unit property) for each $v \in V$ we have $1v = v$,
- (associativity of scalar multiplication) for all $v \in V$ and all $r, s \in \mathbb{R}$ we have $(rs)v = r(sv)$,

Vector spaces

Definition (Vector space)

- (distributivity of scalar multiplication over vector addition) for all $u, v \in V$ and all $r \in \mathbb{R}$ we have that $r(u + v) = ru + rv$, and
- (distributivity of scalar multiplication over scalar addition) for all $v \in V$ and all $r, s \in \mathbb{R}$ we have that $(r + s)v = rv + sv$.

Vector spaces

Show that

$$\{ (x, y) \in \mathbb{R}^2 \mid (x, y) \cdot (3, -1) = 0 \}$$

is closed under addition.