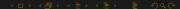
Math 2130 Linear Algebra Week 4 Vector spaces

Charlotte Aten

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# Today's topics

Vector spaces

- We have seen many different collections of objects, such as vectors in  $\mathbb{R}^n$  and various sized matrices, which can be added as well as multiplied by scalars.
- At this point it may seem reasonable that the arithmetic of matrices should be thought of as some kind of generalization of the arithmetic of real numbers themselves.
- The concept of a vector space encompasses this idea, as well as several others we will use throughout the rest of the course.

### Examples of vector spaces

- The vectors in  $\mathbb{R}^n$  where  $(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1 + \ldots, x_n + y_n)$  and  $s(x_1, \ldots, x_n) = (sx_1, \ldots, sx_n)$ .
- The numbers in  $\mathbb R$  with the usual addition and multiplication. Note that  $\mathbb R = \mathbb R^1.$
- The matrices  $\mathrm{Mat}_{m \times n}$  of size  $m \times n$  where addition and scalar multiplication are done componentwise.

### Examples of vector spaces

- The set of all functions  $f: \mathbb{R} \to \mathbb{R}$  where (f+g)(x) = f(x) + g(x) and (sf)(x) = s(f(x)).
- The set of all functions  $f: \mathbb{R}^n \to \mathbb{R}$  where  $(f+g)(x_1,\ldots,x_n) = f(x_1,\ldots,x_n) + g(x_1,\ldots,x_n)$  and  $(sf)(x_1,\ldots,x_n) = s(f(x_1,\ldots,x_n)).$

### Examples of vector spaces

- The set of all quadratic polynomials  $\mathcal{P}_2 = \left\{ \begin{array}{l} ax^2 + bx + c \mid a,b,c \in \mathbb{R} \end{array} \right\} \text{ with } \\ (a_1x^2 + b_1x + c_1) + (a_2x^2 + b_2x + c_2) = \\ (a_1 + a_2)x^2 + (b_1 + b_2)x + (c_1 + c_2) \text{ and } \\ s(ax^2 + bx + c) = (sa)x^2 + (sb)x + (sc). \end{aligned}$
- The set of all degree at most n polynomials, with addition and scalar multiplication defined similarly. This vector space is denoted by  $\mathcal{P}_n$ .

#### Definition (Vector space)

Let V be a set (whose members are called  $\emph{vectors}$ ) on which addition and scalar multiplication are defined. We say that V is a  $\emph{vector space}$  when

- closure under addition) for each pair of vectors u and v in V we have that  $u+v\in V$ ,
- (closure under scalar multiplication) for each vector  $v \in V$  and each scalar  $s \in \mathbb{R}$  we have that  $sv \in V$ ,
- (commutativity of addition) for all  $u, v \in V$  we have that u + v = v + u,

#### Definition (Vector space)

- associativity of addition) for all  $u, v, w \in V$  we have that u + (v + w) = (u + v) + w,
- (zero vector) there is a vector  $0 \in V$  such that for any  $v \in V$  we have v + 0 = v,
- (additive inverse) for each  $v \in V$  there is a vector  $-v \in V$  such that v + (-v) = 0,
- (unit property) for each  $v \in V$  we have 1v = v,
- associativity of scalar multiplication) for all  $v \in V$  and all  $r, s \in \mathbb{R}$  we have (rs)v = r(sv),

#### Definition (Vector space)

- (distributivity of scalar multiplication over vector addition) for all  $u,v\in V$  and all  $r\in \mathbb{R}$  we have that r(u+v)=ru+rv, and
- (distributivity of scalar multiplication over scalar addition) for all  $v \in V$  and all  $r, s \in \mathbb{R}$  we have that (r+s)v = rv + sv.

Show that

$$\{(x,y) \in \mathbb{R}^2 \mid (x,y) \cdot (3,-1) = 0\}$$

is closed under addition.