Math 2130
Linear Algebra
Week 3
Vector spaces

Charlotte Aten

2025 September 12

Today's topics

1. Vector spaces

Vector spaces

Definition

Given variables/numbers/matrices $u_1, \ldots, u_n$ and scalars $c_1, \ldots, c_n \in \mathbb{R}$ we say that

$$c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$$

is a *linear combination* of $u_1, \ldots, u_n$.

- A system of linear equations is asking us to solve something like $xu_1 + yu_2 + zu_3 = w$ where u_1, u_2, u_3, and w are vectors.

Vector spaces

- We have seen many different collections of objects, such as vectors in $\mathbb{R}^n$ and various sized matrices, which can be added as well as multiplied by scalars.
- At this point it may seem reasonable that the arithmetic of matrices should be thought of as some kind of generalization of the arithmetic of real numbers themselves.
- The concept of a vector space encompasses this idea, as well as several others we will use throughout the rest of the course.

Examples of vector spaces

- The vectors in $\mathbb{R}^n$ where $(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1 +, \ldots, x_n + y_n)$ and $s(x_1, \ldots, x_n) = (sx_1, \ldots, sx_n)$.
- The numbers in $\mathbb{R}$ with the usual addition and multiplication. Note that $\mathbb{R} = \mathbb{R}^1$.
- The matrices $\mathrm{Mat}_{m \times n}$ of size $m \times n$ where addition and scalar multiplication are done componentwise.

Examples of vector spaces

- The set of vectors $\{ (x, y) \mid 2x + 3y = 0 \}$ with the usual addition and scalar multiplication from $\mathbb{R}^2$.
- The set of vectors $\{ (x, y, z) \mid (x, y, z) \cdot (1, -1, 4) = 0 \}$ with the usual addition and scalar multiplication from $\mathbb{R}^3$.

Examples of vector spaces

- The set of all functions $f: \mathbb{R} \to \mathbb{R}$ where $(f+g)(x) = f(x) + g(x)$ and $(sf)(x) = s(f(x))$.