Math 2130
Linear Algebra
Week 14
Eigenvectors and eigenvalues

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Today's topics

1. Eigenvectors and eigenvalues

Eigenvectors and eigenvalues

- We find the eigenvalues and corresponding eigenvectors for

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}.$$

- The characteristic polynomial of A is $\lambda^2 - 4\lambda + 3$.

Eigenvectors and eigenvalues

- We saw last time that the eigenvalues of a homomorphism (or matrix) are the roots of its characteristic polynomial.
- For the matrix A, we see that our eigenvalues are the roots of $\lambda^2 - 4\lambda + 3$, which are 3 and 1.

Eigenvectors and eigenvalues

- Now we find the eigenvectors for A.
- Let $v = (v_1, v_2)$.
- Those vectors which satisfy $Av = 3v$ have

$$\begin{aligned} v_1 + 2v_2 &= 3v_1 \\ 3v_2 &= 3v_2. \end{aligned}$$

- Solving the system, we see that $v_2 = v_1$, so the eigenvectors for A with eigenvalue 3 are those of the form (x, x) where $x \neq 0$.

Eigenvectors and eigenvalues

- Again, let $v = (v_1, v_2)$.
- Those vectors which satisfy $Av = 1v$ have

$$\begin{aligned} v_1 + 2v_2 &= v_1 \\ 3v_2 &= v_2. \end{aligned}$$

- Solving the system, we see that $v_2 = 0$, so the eigenvectors for A with eigenvalue 1 are those of the form $(x, 0)$ where $x \neq 0$.

Eigenvectors and eigenvalues

- We find the eigenvalues and corresponding eigenvectors for

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

- The characteristic polynomial of A is $\lambda^2 - 4\lambda + 4$.
- We find that A has only one eigenvalue, 2.

Eigenvectors and eigenvalues

- Let $v = (v_1, v_2)$.
- Those vectors which satisfy $Av = 2v$ have

$$
\begin{aligned}
2v_1 &= 2v_1 \\
2v_2 &= 2v_2.
\end{aligned}
$$

- Solving the system, we see that any nonzero vector (x, y) is an eigenvector for A with eigenvalue 2.

Eigenvectors and eigenvalues

- We find the eigenvalues and corresponding eigenvectors for

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

- The characteristic polynomial of A is $\lambda^2 + 1$.
- We find that A has no real eigenvalues, and hence no real eigenvectors.
- Geometrically, we see that there is no direction where A scales $\mathbb{R}^2$ since A rotates by $\frac{\pi}{2}$ radians.

Eigenvectors and eigenvalues

- If we allow complex numbers, we find that i and $-i$ are the eigenvalues of A.
- Let $v = (v_1, v_2)$.
- Those vectors which satisfy $Av = iv$ have

$$\begin{aligned} -v_2 &= iv_1 \\ v_1 &= iv_2. \end{aligned}$$

- Solving the system, we see that $v_2 = -iv_1$ and thus all vectors of the form $(x, -ix)$ for $x \neq 0$ are eigenvectors for A, if we allow vectors with nonreal entries.

Eigenvectors and eigenvalues

- I won't give you problems like this, but it's worth seeing that one must in general allow complex numbers to find all the eigenvalues and eigenvectors of a real matrix.
- The idea here is that rotation is actually a form of scaling, where the scalar in question is imaginary.

Eigenvectors and eigenvalues

- In general, a matrix may have both real and nonreal eigenvalues.
- Let
$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$
- The characteristic polynomial of A is $(\lambda - 2)(\lambda^2 + 1)$.
- We find that A has one real eigenvalue, 2, and two imaginary eigenvalues, $\pm i$.
- Geometrically, A rotates the xy-plane in $\mathbb{R}^3$ but scales the z axis by a factor of 2.

Eigenvectors and eigenvalues

- We find the eigenvalues and corresponding eigenvectors for
$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$
- The characteristic polynomial of A is $\lambda^2 - 2\lambda + 1$.

Eigenvectors and eigenvalues

- For the matrix A, we see that our eigenvalues are the roots of $\lambda^2 - 2\lambda + 1$.
- The only eigenvalue of A is 1, and it is a root of multiplicity 2 of the characteristic polynomial.

Eigenvectors and eigenvalues

- Now we find the eigenvectors for A.
- Let $v = (v_1, v_2)$.
- Those vectors which satisfy $Av = 1v$ have

$$\begin{aligned} v_1 + 2v_2 &= v_1 \\ v_2 &= v_2. \end{aligned}$$

- Solving the system, we see that $v_2 = 0$, so the eigenvectors for A with eigenvalue 1 are those of the form $(x, 0)$ where $x \neq 0$.

Eigenvectors and eigenvalues

- Note that we only have a 1-dimensional subspace of $\mathbb{R}^2$ which consists of eigenvectors for the eigenvalue 1, even though 1 occurs as a root of the characteristic polynomial with multiplicity 2.