Math 2130
Linear Algebra
Week 14
Eigenvectors and eigenvalues

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Today's topics

1. Eigenvectors and eigenvalues

Eigenvectors and eigenvalues

- Recall our example of a homomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ where $f(x, y) = (x + y, 2x + 2y)$.
- For the basis $B = \{(1, 0), (0, 1)\}$ we have

$$[f]_B^B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

- For the basis $C = \{(1, 2), (1, -1)\}$ we have that

$$[f]_C^C = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}.$$

Eigenvectors and eigenvalues

- The vectors $(1,2)$ and $(1,-1)$ are called *eigenvectors* for f.
- Since

  $$f(1,2) = (3,6) = 3(1,2)$$

  we say that $(1,2)$ is an eigenvector for f with *eigenvalue* 3.
- Since

  $$f(1,-1) = (0,0) = 0(1,-1)$$

  we say that $(1,-1)$ is an eigenvector for f with eigenvalue 0.

Eigenvectors and eigenvalues

- The lines in $\mathbb{R}^2$ spanned by $(1,2)$ and $(1,-1)$ are the basic directions in which f stretches the plane.
- The corresponding eigenvalues measure the amount of stretching (or *scaling*) in that direction.

Eigenvectors and eigenvalues

- Note that most vectors are not eigenvectors for f.
- For example, $f(1,0) = (1,2)$ and $(1,2) = k(1,0)$ is never true for any value of k.

Eigenvectors and eigenvalues

Definition

Given a homomorphism $f\colon V \to V$, we say that $v \in V$ is an *eigenvector* for f with *eigenvalue* λ when $f(v) = \lambda v$ and $v \neq 0$.

- If $A \in \mathrm{Mat}_{n\times n}$ then we speak of eigenvectors and eigenvalues for $f_A\colon \mathbb{R}^n \to \mathbb{R}^n$ as eigenvectors and eigenvalues for A.
- This is just like how we can define the image or kernel of A as the image or kernel of the homomorphism f_A.

Eigenvectors and eigenvalues

- I'll show how to find the eigenvalues for

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}.$$

- Note that if $Av = \lambda v$ then

$$Av = \lambda I v$$

so

$$(\lambda I - A)v = 0.$$

Eigenvectors and eigenvalues

- Thus, v is an eigenvector for A with eigenvalue λ exactly when $v \in \mathrm{Ker}(\lambda I - A)$ (and $v \neq 0$).
- This means that $\lambda I - A$ must have a nonzero kernel in order for λ to be an eigenvalue for A.
- Thus, the eigenvalues of A are those λ with

$$\det(\lambda I - A) = 0.$$

Eigenvectors and eigenvalues

Definition

The *characteristic polynomial* of a matrix A is $\det(\lambda I - A)$.

- We can define the characteristic polynomial of a homomorphism $f\colon V \to V$ to be the characteristic polynomial of $[f]_B^B$ where B is any basis for V. (They all give the same polynomial.)

Eigenvectors and eigenvalues

- When $f(x, y) = (x + y, 2x + 2y)$ we have that the characteristic polynomial of f is $\lambda^2 - 3\lambda$.
- The characteristic polynomial of

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

  is $\lambda^2 - 4\lambda + 3$.

Eigenvectors and eigenvalues

- Our previous reasoning says that the eigenvalues of a homomorphism (or matrix) are the roots of its characteristic polynomial.
- This checks out for $f(x, y) = (x + y, 2x + 2y)$ because we found eigenvectors with eigenvalues 3 and 0, which are the roots of $\lambda^2 - 3\lambda$.
- For the matrix A, we see that our eigenvalues are the roots of $\lambda^2 - 4\lambda + 3$, which are 3 and 1.

Eigenvectors and eigenvalues

- Now we find the eigenvectors for A.
- Let $v = (v_1, v_2)$.
- Those vectors which satisfy $Av = 3v$ have

$$\begin{aligned} v_1 + 2v_2 &= 3v_1 \\ 3v_2 &= 3v_2. \end{aligned}$$

- Solving the system, we see that $v_2 = v_1$, so the eigenvectors for A with eigenvalue 3 are those of the form (x, x) where $x \neq 0$.