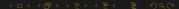
# Math 2130 Linear Algebra Week 14 Eigenvectors and eigenvalues

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# Today's topics

- Recall our example of a homomorphism  $f: \mathbb{R}^2 \to \mathbb{R}^2$  where f(x,y) = (x+y,2x+2y).
- For the basis  $B = \{(1,0),(0,1)\}$  we have

$$[f]_B^B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

■ For the basis  $C = \{(1,2), (1,-1)\}$  we have that

$$[f]_C^C = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}.$$

- The vectors (1,2) and (1,-1) are called *eigenvectors* for f.
- Since

$$f(1,2) = (3,6) = 3(1,2)$$

we say that (1,2) is an eigenvector for f with eigenvalue 3.

Since

$$f(1,-1) = (0,0) = 0(1,-1)$$

we say that (1,-1) is an eigenvector for f with eigenvalue 0.



- The lines in  $\mathbb{R}^2$  spanned by (1,2) and (1,-1) are the basic directions in which f stretches the plane.
- The corresponding eigenvalues measure the amount of stretching (or *scaling*) in that direction.

- Note that most vectors are not eigenvectors for f.
- For example, f(1,0) = (1,2) and (1,2) = k(1,0) is never true for any value of k.

#### Definition

Given a homomorphism  $f \colon V \to V$ , we say that  $v \in V$  is an eigenvector for f with eigenvalue  $\lambda$  when  $f(v) = \lambda v$  and  $v \neq 0$ .

- If  $A \in \operatorname{Mat}_{n \times n}$  then we speak of eigenvectors and eigenvalues for  $f_A : \mathbb{R}^n \to \mathbb{R}^n$  as eigenvectors and eigenvalues for A.
- This is just like how we can define the image or kernel of A as the image or kernel of the homomorphism  $f_A$ .

I'll show how to find the eigenvalues for

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}.$$

■ Note that if  $Av = \lambda v$  then

$$Av = \lambda Iv$$

so

$$(\lambda I - A)v = 0.$$

- Thus, v is an eigenvector for A with eigenvalue  $\lambda$  exactly when  $v \in \operatorname{Ker}(\lambda I A)$  (and  $v \neq 0$ ).
- This means that  $\lambda I A$  must have a nonzero kernel in order for  $\lambda$  to be an eigenvalue for A.
- Thus, the eigenvalues of A are those  $\lambda$  with

$$\det(\lambda I - A) = 0.$$

#### Definition

The characteristic polynomial of a matrix A is  $\det(\lambda I - A)$ .

■ We can define the characteristic polynomial of a homomorphism  $f\colon V\to V$  to be the characteristic polynomial of  $[f]_B^B$  where B is any basis for V. (They all give the same polynomial.)

- When f(x,y) = (x+y,2x+2y) we have that the characteristic polynomial of f is  $\lambda^2 3\lambda$ .
- The characteristic polynomial of

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

is 
$$\lambda^2 - 4\lambda + 3$$
.

- Our previous reasoning says that the eigenvalues of a homomorphism (or matrix) are the roots of its characteristic polynomial.
- This checks out for f(x,y)=(x+y,2x+2y) because we found eigenvectors with eigenvalues 3 and 0, which are the roots of  $\lambda^2-3\lambda$ .
- For the matrix A, we see that our eigenvalues are the roots of  $\lambda^2 4\lambda + 3$ , which are 3 and 1.

- $\blacksquare$  Now we find the eigenvectors for A.
- Let  $v = (v_1, v_2)$ .
- Those vectors which satisfy Av = 3v have

$$v_1 + 2v_2 = 3v_1$$
$$3v_2 = 3v_2.$$

■ Solving the system, we see that  $v_2 = v_1$ , so the eigenvectors for A with eigenvalue 3 are those of the form (x,x) where  $x \neq 0$ .