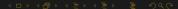
Math 2130 Linear Algebra Week 13 Determinants

Charlotte Aten

2025 November 21



Today's topics

- The permutation expansion
- 2 The cofactor method

■ I will remind you how to use the multilinear property to obtain the formula for the 3×3 determinant, as we saw last time.

Definition

An *n*-permutation is a bijection Φ : $\{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$.

Definition

Given an n-permutation Φ the permutation matrix for Φ is the $n \times n$ matrix M_{Φ} whose i^{th} column is $e_{\Phi(i)}$.

■ This is the same as saying that M_Φ is the change of basis matrix $[\mathrm{id}]_B^C$ where $B=\{b_1,\ldots,b_n\}$ and $C=\left\{b_{\Phi(1)},\ldots,b_{\Phi(n)}\right\}$.

Definition

The sign of an n-permutation Φ is $det(M_{\Phi})$. We denote the sign of Φ by $sgn(\Phi)$.

- We have that $\operatorname{sgn}(\operatorname{id}) = 1$ where id is the trivial permutation with $\operatorname{id}(x) = x$ for any $x \in \{1, \dots, n\}$.
- We have that ${\rm sgn}(\Phi)=\pm 1$ depending on whether it takes an even or odd number of column swaps to get from $I_n=M_{\rm id}$ to $M_\Phi.$

Definition

The permutation expansion for the $n \times n$ determinant is

$$\det(A) = \sum_{\text{permutations } \Phi} \operatorname{sgn}(\Phi) \prod_{i=1}^{n} a_{i,\Phi(i)}.$$

The cofactor method

- When we use the multilinear property to express the determinant of a matrix in terms of determinants of matrices with more zeroes, we don't need to do every single row.
- Notice what happens if we just expand a 3×3 matrix along its first row.

The cofactor method

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ d & e & f \\ g & h & i \end{vmatrix}$$
$$= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

The cofactor method

- We can use the same method on any row or column to write an $n \times n$ determinant as a linear combination of $(n-1) \times (n-1)$ determinants.
- We must include a factor of -1 for each term where the corresponding entry in the original matrix is a_{ij} with i+j an odd number.