Math 2130
Linear Algebra
Week 13
Determinants

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Today's topics

1. The permutation expansion
2. The cofactor method

The permutation expansion

- I will remind you how to use the multilinear property to obtain the formula for the 3×3 determinant, as we saw last time.

The permutation expansion

Definition

An *n-permutation* is a bijection $\Phi\colon \{1,\ldots,n\} \to \{1,\ldots,n\}$.

The permutation expansion

Definition

Given an n-permutation Φ the *permutation matrix* for Φ is the $n \times n$ matrix M_Φ whose i^{th} column is $e_{\Phi(i)}$.

- This is the same as saying that M_Φ is the change of basis matrix $[\mathrm{id}]_B^C$ where $B = \{b_1, \ldots, b_n\}$ and $C = \left\{b_{\Phi(1)}, \ldots, b_{\Phi(n)}\right\}$.

The permutation expansion

Definition

The *sign* of an n-permutation Φ is $\det(M_\Phi)$. We denote the sign of Φ by $\operatorname{sgn}(\Phi)$.

- We have that $\operatorname{sgn}(\mathrm{id}) = 1$ where id is the trivial permutation with $\mathrm{id}(x) = x$ for any $x \in \{1, \ldots, n\}$.
- We have that $\operatorname{sgn}(\Phi) = \pm 1$ depending on whether it takes an even or odd number of column swaps to get from $I_n = M_{\mathrm{id}}$ to M_Φ.

The permutation expansion

Definition

The *permutation expansion* for the $n \times n$ determinant is

$$\det(A) = \sum_{\text{permutations } \Phi} \operatorname{sgn}(\Phi) \prod_{i=1}^{n} a_{i,\Phi(i)}.$$

The cofactor method

- When we use the multilinear property to express the determinant of a matrix in terms of determinants of matrices with more zeroes, we don't need to do every single row.
- Notice what happens if we just expand a 3×3 matrix along its first row.

The cofactor method

$$\begin{aligned}\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= \begin{vmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ d & e & f \\ g & h & i \end{vmatrix} \\ &= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}\end{aligned}$$

The cofactor method

- We can use the same method on any row or column to write an $n \times n$ determinant as a linear combination of $(n-1) \times (n-1)$ determinants.
- We must include a factor of -1 for each term where the corresponding entry in the original matrix is a_{ij} with $i + j$ an odd number.