Math 2130
Linear Algebra
Week 13
Determinants

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2025 November 19

Today's topics

1. Properties of determinants
2. The permutation expansion

Properties of determinants

Definition

An $n \times n$ *determinant* is a function $\det\colon \mathrm{Mat}_{n\times n} \to \mathbb{R}$ such that

1. $\det(\rho_1, \ldots, k\rho_i + \rho_j, \ldots, \rho_n) = \det(\rho_1, \ldots, \rho_j, \ldots, \rho_n)$ when $i \neq j$,
2. $\det(\rho_1, \ldots, \rho_j, \ldots, \rho_i, \ldots, \rho_n) = -\det(\rho_1, \ldots, \rho_i, \ldots, \rho_j, \ldots, \rho_n)$ when $i \neq j$,
3. $\det(\rho_1, \ldots, k\rho_i, \ldots, \rho_n) = k\det(\rho_1, \ldots, \rho_i, \ldots, \rho_n)$ for any scalar k, and
4. $\det(I_n) = 1$.

Properties of determinants

- We saw last time that these properties uniquely describe a function $\det\colon \mathrm{Mat}_{n\times n} \to \mathbb{R}$, which we call the determinant.
- We also saw that $\det(A) \neq 0$ exactly when A is nonsingular.
- I showed how to compute the determinant of a matrix using the properties from the definition.
- I'll do this again with the matrix $\begin{bmatrix} 0 & 2 \\ 3 & 8 \end{bmatrix}$ to remind you.

Properties of determinants

- Similarly, the determinant of a matrix with all zeroes above (or below) the main diagonal is the product of the diagonal entries.

The permutation expansion

Definition

Given a vector space V we say that a map $f\colon V^n \to \mathbb{R}$ is *multilinear* when for all $v, w \in V$ and all $s \in \mathbb{R}$ we have

1. $f(\rho_1, \dots, v + w, \dots, \rho_n) = f(\rho_1, \dots, v, \dots, \rho_n) + f(\rho_1, \dots, w, \dots, \rho_n)$ and
2. $f(\rho_1, \dots, sv, \dots, \rho_n) = sf(\rho_1, \dots, v, \dots, \rho_n)$.

- Determinants are multilinear.

The permutation expansion

Theorem

A function $\det\colon \mathrm{Mat}_{n\times n} \to \mathbb{R}$ *is a determinant function exactly when it*

1. *is multilinear,*
2. $\det(\rho_1, \ldots, \rho_j, \ldots, \rho_i, \ldots, \rho_n) = -\det(\rho_1, \ldots, \rho_i, \ldots, \rho_j, \ldots, \rho_n)$ *when* $i \neq j$ *(i.e.* $\det$ *is alternating), and*
3. *it has* $\det(I_n) = 1$.

The permutation expansion

- I will use the multilinear property to obtain the formulas for the 2×2 and 3×3 determinant.