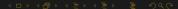
Math 2130 Linear Algebra Week 13 Determinants

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2025 November 19



Today's topics

- Properties of determinants
- The permutation expansion

Properties of determinants

Definition

An $n \times n$ determinant is a function $\det: \operatorname{Mat}_{n \times n} \to \mathbb{R}$ such that

- $\det(
 ho_1,\ldots,k
 ho_i+
 ho_j,\ldots,
 ho_n)=\det(
 ho_1,\ldots,
 ho_j,\ldots,
 ho_n)$ when i
 eq j,
- $\det(\rho_1, \dots, \rho_j, \dots, \rho_i, \dots, \rho_n) = \\ -\det(\rho_1, \dots, \rho_i, \dots, \rho_j, \dots, \rho_n) \text{ when } i \neq j,$
- $\det(\rho_1,\ldots,k\rho_i,\ldots,\rho_n)=k\det(\rho_1,\ldots,\rho_i,\ldots,\rho_n)$ for any scalar k, and
- $\det(I_n) = 1.$

Properties of determinants

- We saw last time that these properties uniquely describe a function $\det: \operatorname{Mat}_{n \times n} \to \mathbb{R}$, which we call the determinant.
- We also saw that $det(A) \neq 0$ exactly when A is nonsingular.
- I showed how to compute the determinant of a matrix using the properties from the definition.
- I'll do this again with the matrix $\begin{bmatrix} 0 & 2 \\ 3 & 8 \end{bmatrix}$ to remind you.

Properties of determinants

■ Similarly, the determinant of a matrix with all zeroes above (or below) the main diagonal is the product of the diagonal entries.

The permutation expansion

Definition

Given a vector space V we say that a map $f \colon V^n \to \mathbb{R}$ is multilinear when for all $v, w \in V$ and all $s \in \mathbb{R}$ we have

- $f(\rho_1,\ldots,v+w,\ldots,\rho_n)=\\f(\rho_1,\ldots,v,\ldots,\rho_n)+f(\rho_1,\ldots,w,\ldots,\rho_n) \text{ and }$
- $f(\rho_1,\ldots,sv,\ldots,\rho_n) = sf(\rho_1,\ldots,v,\ldots,\rho_n).$
- Determinants are multilinear.

The permutation expansion

Theorem

A function $\det: \operatorname{Mat}_{n \times n} \to \mathbb{R}$ is a determinant function exactly when it

- is multilinear,
- $\det(\rho_1,\ldots,\rho_j,\ldots,\rho_i,\ldots,\rho_n) = \\ -\det(\rho_1,\ldots,\rho_i,\ldots,\rho_j,\ldots,\rho_n) \text{ when } i\neq j \text{ (i.e. det is alternating), and}$
- $\operatorname{lt} \operatorname{has} \det(I_n) = 1.$

The permutation expansion

■ I will use the multilinear property to obtain the formulas for the 2×2 and 3×3 determinant.