Math 2130
Linear Algebra
Week 13
Determinants

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Today's topics

1. Exploration
2. Properties of determinants

Exploration

- A 1×1 matrix $[a]$ is nonsingular if and only if $a \neq 0$.
- A 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is nonsingular if and only if $ad - bc \neq 0$.
- One can show that a 3×3 matrix

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

is nonsingular if and only if

$$aei + bfg + cdh - hfa - idb - gec \neq 0.$$

Exploration

- It seems that there may be, for each n, a *determinant* function

$$\det_n\colon \mathrm{Mat}_{n\times n} \to \mathbb{R}$$

where $\det_n(A) \neq 0$ exactly when A is nonsingular.

Exploration

- To find what such a formula would look like, we check how the known determinant functions behave with respect to the elementary row operations.

Exploration

- Adding a multiple of one row to another doesn't change any of the potential determinant formulas we found before.
- Swapping two rows negates the previously given formulas.
- Multiplying a row by a scalar multiplies the value of the formula by the same scalar.

Properties of determinants

Definition

An $n \times n$ *determinant* is a function $\det\colon \mathrm{Mat}_{n\times n} \to \mathbb{R}$ such that

1. $\det(\rho_1, \ldots, k\rho_i + \rho_j, \ldots, \rho_n) = \det(\rho_1, \ldots, \rho_j, \ldots, \rho_n)$ when $i \neq j$,
2. $\det(\rho_1, \ldots, \rho_j, \ldots, \rho_i, \ldots, \rho_n) = -\det(\rho_1, \ldots, \rho_i, \ldots, \rho_j, \ldots, \rho_n)$ when $i \neq j$,
3. $\det(\rho_1, \ldots, k\rho_i, \ldots, \rho_n) = k\det(\rho_1, \ldots, \rho_i, \ldots, \rho_n)$ for any scalar k, and
4. $\det(I_n) = 1$.