Math 2130
Linear Algebra
Week 12
Changing representations

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Today's topics

1. Changing representations

Changing representations

- The *identity homomorphism* for a vector space V is the function $\mathrm{id}\colon V \to V$ given by $\mathrm{id}(v) = v$.
- Given any basis B for V, we have that $[\mathrm{id}]_B^B = I_n$ where $n = \dim(V)$.
- We sometimes write id_V to emphasize that we are talking about the identity homomorphism for the vector space V, as opposed to the identity homomorphism for some other vector space.
- Note that if $h\colon V \to W$ is a homomorphism then $\mathrm{id}_W \circ h = h = h \circ \mathrm{id}_V$.
- In particular, $\mathrm{id}_V \circ \mathrm{id}_V = \mathrm{id}_V$.

Changing representations

Proposition

Given a homomorphism $h \colon V \to W$, *bases* B_1 *and* B_2 *for* V, *and bases* C_1 *and* C_2 *for* W *we have that*

$$[h]^{C_1}_{B_1}[\mathrm{id}_V]^{B_1}_{B_2} = [h]^{C_1}_{B_2}$$

and

$$[\mathrm{id}_W]^{C_2}_{C_1}[h]^{C_1}_{B_1} = [h]^{C_2}_{B_1}.$$

Changing representations

Definition (Change of basis matrix)

Given a vector space V and two bases B and C, the *change of basis* matrix from B to C is

$$[\mathrm{id}]_B^C$$

where $\mathrm{id}\colon V \to V$ is the identity homomorphism for V.

Changing representations

Proposition

A square matrix is a change of basis matrix if and only if it is nonsingular.

- This is because

$$[\mathrm{id}]_C^B[\mathrm{id}]_B^C = [\mathrm{id} \circ \mathrm{id}]_B^B = [\mathrm{id}]_B^B = I_n.$$

- In particular, every elementary matrix is a change of basis matrix.
- Note that $[\mathrm{id}]_C^B = ([\mathrm{id}]_B^C)^{-1}$.

Changing representations

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x,y) = (x+y, 2x+2y)$ and let $B = \{(1,0),(0,1)\}$ and $C = \{(1,2),(1,-1)\}$ be basis for $\mathbb{R}^2$. Use the change of basis matrices $[\mathrm{id}]_B^C$ and $[\mathrm{id}]_C^B$ to compute $[f]_C^C$ given that $[f]_B^B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.