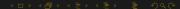
Math 2130 Linear Algebra Week 12 Representing homomorphisms

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Today's topics

Representing homomorphisms

Representing homomorphisms

Definition

Given finite-dimensional vector spaces V and W with bases $B=\{v_1,v_2,\ldots,v_n\}$ and $C=\{w_1,w_2,\ldots,w_m\}$, respectively, and a homomorphisms $f\colon V\to W$ the matrix representation of f relative to the bases B and C is the $m\times n$ matrix

$$[f]_B^C = [[f(v_1)]_C \quad [f(v_2)]_C \quad \cdots \quad [f(v_n)]_C]$$

where

$$\begin{bmatrix} x \end{bmatrix}_C = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^T$$

when $x = x_1 w_1 + x_2 w_2 + \cdots + x_m w_m$.



Representing homomorphisms

Proposition

Given vector spaces U, V, and W, homomorphisms $f\colon U\to V$ and $g\colon V\to W$, and bases $B_U=\{u_1,\ldots,u_n\}$, $B_V=\{v_1,\ldots,v_m\}$, and $B_W=\{w_1,\ldots,w_k\}$ we have that $[g\circ f]_{B^{**}}^{B_W}=[g]_{B^{**}}^{B_W}[f]_{B^{**}}^{B_V}.$

Any matrix represents a homomorphism

Theorem

Any matrix represents a homomorphism between vector spaces of appropriate dimensions, with respect to any pair of bases.

Any matrix represents a homomorphism

Theorem

The rank of a matrix equals the rank (dimension of the image) of any homomorphism it represents.

Any matrix represents a homomorphism

Corollary

Given a homomorphism h represented by a matrix $H = [h]_B^C$ we have that h is surjective if and only if the rank of H equals the number of its rows and h is injective if and only if its rank equals the number of its columns.