Math 2130
Linear Algebra
Week 12
Representing homomorphisms

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Today's topics

1. Representing homomorphisms

Representing homomorphisms

Definition

Given finite-dimensional vector spaces V and W with bases $B = \{v_1, v_2, \ldots, v_n\}$ and $C = \{w_1, w_2, \ldots, w_m\}$, respectively, and a homomorphisms $f\colon V \to W$ the *matrix representation* of f relative to the bases B and C is the $m \times n$ matrix

$$[f]_B^C = \left[\left[f(v_1)\right]_C \quad \left[f(v_2)\right]_C \quad \cdots \quad \left[f(v_n)\right]_C \right]$$

where

$$\left[x\right]_C = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^T$$

when $x = x_1 w_1 + x_2 w_2 + \cdots + x_m w_m$.

Representing homomorphisms

Proposition

Given vector spaces U, V, and W, homomorphisms $f\colon U \to V$ and $g\colon V \to W$, and bases $B_U = \{u_1, \ldots, u_n\}$, $B_V = \{v_1, \ldots, v_m\}$, and $B_W = \{w_1, \ldots, w_k\}$ we have that

$$[g \circ f]_{B_U}^{B_W} = [g]_{B_V}^{B_W} [f]_{B_U}^{B_V}.$$

Any matrix represents a homomorphism

Theorem

Any matrix represents a homomorphism between vector spaces of appropriate dimensions, with respect to any pair of bases.

Any matrix represents a homomorphism

Theorem

The rank of a matrix equals the rank (dimension of the image) of any homomorphism it represents.

Any matrix represents a homomorphism

Corollary

Given a homomorphism h *represented by a matrix* $H = [h]_B^C$ *we have that* h *is surjective if and only if the rank of* H *equals the number of its rows and* h *is injective if and only if its rank equals the number of its columns.*