Math 2130
Linear Algebra
Week 11
The Rank-Nullity Theorem

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Today's topics

1. Bases for images and kernels
2. The Rank-Nullity Theorem

Bases for images and kernels

- If we have a homomorphism $f\colon V \to W$ and a basis $\{b_1, \ldots, b_n\}$ for V, we can find a basis for $\mathrm{Ker}(f)$ by solving for the values of $c_1, \ldots, c_n$ that make

$$f(c_1 b_1 + \cdots + c_n b_n) = 0.$$

- We can find a basis for $\mathrm{Im}(f)$ by noting that $\mathrm{Im}(f) = \mathrm{Span}(\{f(b_1), \ldots, f(b_n)\})$.

Bases for images and kernels

- Let $f\colon V \to W$ be a homomorphism.
- Note that if $\mathrm{Ker}(f) = \{0\}$ then f is injective, while if $\mathrm{Ker}(f) = V$ then $f(v) = 0$.
- Similarly, if $\mathrm{Im}(f) = W$ then f is surjective, while if $\mathrm{Im}(f) = \{0\}$ then $f(v) = 0$.

Bases for images and kernels

- Consider the homomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x + y, 2x + 2y)$.
- A basis for $\mathrm{Ker}(f)$ is $\{(1, -1)\}$.
- A basis for $\mathrm{Im}(f)$ is $\{(1, 2)\}$.

The Rank-Nullity Theorem

Theorem (The Rank-Nullity Theorem)

Given a homomorphism $f\colon V \to W$ *where* V *is finite-dimensional we have that*

$$\dim(\mathrm{Im}(f)) + \dim(\mathrm{Ker}(f)) = \dim(V).$$

- We can also think about this as a statement about the rank of a matrix A by considering the homomorphism f_A given by $f_A(v) = Av$.

The Rank-Nullity Theorem

- Our previous example shows that, in general, a basis isn't split up into a basis for the kernel and vectors that map to a basis for the image.
- Neither $(1, 0)$ or $(0, 1)$ is in the kernel of f for that example, but $\mathrm{Ker}(f)$ is 1-dimensional.

The Rank-Nullity Theorem

- Let's consider a bigger example.
- Take $f\colon \mathbb{R}^4 \to \mathbb{R}^4$ to be given by

$$f(a,b,c,d) = (a+b+c, a-b-d, 2a+c-d, 3a+b+2c-d).$$

- A basis for $\mathrm{Ker}(f)$ is

$$\left\{ \left(\frac{-1}{2}, \frac{-1}{2}, 1, 0 \right), \left(\frac{1}{2}, \frac{-1}{2}, 0, 1 \right) \right\}.$$

- A basis for $\mathrm{Im}(f)$ is

$$\{(1,1,2,3),(1,-1,0,1)\}.$$