Math 2130 Linear Algebra Week 11 Bases for images and kernels

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Today's topics

- Images
- Bases for images and kernels

Definition

Given a homomorphism $f \colon V \to W$ the image (or range) of f is

$$\operatorname{Im}(f) = \left\{ f(v) \in W \mid v \in V \right\}.$$

■ Note that Im(f) is a set of vectors from W.

- Consider the homomorphism $f: \mathbb{R}^2 \to \mathbb{R}^2$ given by f(x,y) = (x,0).
- We have that

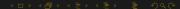
$$\operatorname{Im}(f) = \left\{ f(x, y) \in \mathbb{R}^2 \mid (x, y) \in \mathbb{R}^2 \right\}$$
$$= \left\{ (x, 0) \in \mathbb{R}^2 \mid x, y \in \mathbb{R} \right\}$$
$$= \left\{ (x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R} \right\}.$$

 \blacksquare This is all the stuff we obtain by plugging vectors into f.

- Consider the homomorphism $f: \mathbb{R}^2 \to \mathbb{R}^2$ given by f(x,y) = (x+y,2x+2y).
- We have that

$$\operatorname{Im}(f) = \left\{ f(x, y) \in \mathbb{R}^2 \mid (x, y) \in \mathbb{R}^2 \right\}$$
$$= \left\{ (x + y, 2x + 2y) \in \mathbb{R}^2 \mid x, y \in \mathbb{R} \right\}$$
$$= \left\{ (x, 2x) \in \mathbb{R}^2 \mid x \in \mathbb{R} \right\}.$$

Note that this is again all the stuff we obtain by plugging vectors into f.



Proposition

Given any homomorphism $f \colon V \to W$, we have that $\mathrm{Im}(f)$ is a subspace of W.

■ We can show this by using the Subspace Test.

Let
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
.

lacksquare The homomorphism $f_A\colon \mathbb{R}^2 o \mathbb{R}^2$ given by $f_A(v)=Av$ has

$$\operatorname{Im}(f_A) = \left\{ f_A(x, y) \in \mathbb{R}^2 \mid (x, y) \in \mathbb{R}^2 \right\}$$

$$= \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x, y \in \mathbb{R} \right\}$$

$$= \left\{ x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 4 \end{bmatrix} \in \mathbb{R}^2 \mid x, y \in \mathbb{R}^2 \right\}$$

$$= \mathbb{R}^2.$$

■ Note that Im(A) = Col(A).

- In general, $Im(f_A) = Col(A)$ for any matrix A.
- We call $\dim(\operatorname{Im}(f))$ the *rank* of f for any homomorphism.
- Note that this definition makes the rank of f_A the rank of the matrix A.
- Similarly, we call $\dim(\operatorname{Ker}(f))$ the *nullity* of f.

- Let's compute some ranks and nullities by finding bases.
- If we have a homomorphism $f \colon V \to W$ and a basis $\{b_1, \ldots, b_n\}$ for V, we can find a basis for $\operatorname{Ker}(f)$ by solving for the values of c_1, \ldots, c_n that make

$$f(c_1b_1+\cdots+c_nb_n)=0.$$

■ We can find a basis for Im(f) by noting that $Im(f) = Span(\{f(b_1), \dots, f(b_n)\}).$

- Consider the homomorphism $f: \mathbb{R}^2 \to \mathbb{R}^2$ given by f(x,y) = (x,0).
- A basis for Ker(f) is $\{(0,1)\}$.
- A basis for Im(f) is $\{(1,0)\}$.

- Consider the homomorphism $f: \mathbb{R}^2 \to \mathbb{R}^2$ given by f(x,y) = (x+y,2x+2y).
- A basis for Ker(f) is $\{(1,-1)\}$.
- A basis for Im(f) is $\{(1,2)\}$.

Kernels

Let
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
.

- In this case $Ker(f_A)$ is trivial.
- A basis for $Im(f_A)$ is $\{(1,3),(2,4)\}$.