Math 2130
Linear Algebra
Week 11
Bases for images and kernels

Charlotte Aten

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Today's topics

1. Images
2. Bases for images and kernels

Images

Definition

Given a homomorphism $f\colon V \to W$ the *image* (or *range*) of f is

$$\operatorname{Im}(f) = \{\, f(v) \in W \mid v \in V \,\}.$$

- Note that $\operatorname{Im}(f)$ is a set of vectors from W.

Images

- Consider the homomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x, 0)$.
- We have that

$$\begin{aligned}
\operatorname{Im}(f) &= \left\{\, f(x,y) \in \mathbb{R}^2 \;\middle|\; (x,y) \in \mathbb{R}^2 \,\right\} \\
&= \left\{\, (x,0) \in \mathbb{R}^2 \;\middle|\; x, y \in \mathbb{R} \,\right\} \\
&= \left\{\, (x,0) \in \mathbb{R}^2 \;\middle|\; x \in \mathbb{R} \,\right\}.
\end{aligned}$$

- This is all the stuff we obtain by plugging vectors into f.

Images

- Consider the homomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x + y, 2x + 2y)$.
- We have that

$$
\begin{aligned}
\operatorname{Im}(f) &= \{\, f(x, y) \in \mathbb{R}^2 \mid (x, y) \in \mathbb{R}^2 \,\} \\
&= \{\, (x + y, 2x + 2y) \in \mathbb{R}^2 \mid x, y \in \mathbb{R} \,\} \\
&= \{\, (x, 2x) \in \mathbb{R}^2 \mid x \in \mathbb{R} \,\}.
\end{aligned}
$$

- Note that this is again all the stuff we obtain by plugging vectors into f.

Images

Proposition

Given any homomorphism $f \colon V \to W$*, we have that* $\mathrm{Im}(f)$ *is a subspace of* W*.*

- We can show this by using the Subspace Test.

Bases for images and kernels

- Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.
- The homomorphism $f_A \colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f_A(v) = Av$ has

$$\begin{aligned} \mathrm{Im}(f_A) &= \left\{ f_A(x, y) \in \mathbb{R}^2 \mid (x, y) \in \mathbb{R}^2 \right\} \\ &= \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x, y \in \mathbb{R} \right\} \\ &= \left\{ x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 4 \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x, y \in \mathbb{R}^2 \right\} \\ &= \mathbb{R}^2. \end{aligned}$$

- Note that $\mathrm{Im}(A) = \mathrm{Col}(A)$.

Bases for images and kernels

- In general, $\mathrm{Im}(f_A) = \mathrm{Col(A)}$ for any matrix A.
- We call $\dim(\mathrm{Im}(f))$ the *rank* of f for any homomorphism.
- Note that this definition makes the rank of f_A the rank of the matrix A.
- Similarly, we call $\dim(\mathrm{Ker}(f))$ the *nullity* of f.

Bases for images and kernels

- Let's compute some ranks and nullities by finding bases.
- If we have a homomorphism $f\colon V \to W$ and a basis $\{b_1, \ldots, b_n\}$ for V, we can find a basis for $\mathrm{Ker}(f)$ by solving for the values of $c_1, \ldots, c_n$ that make

$$f(c_1 b_1 + \cdots + c_n b_n) = 0.$$

- We can find a basis for $\mathrm{Im}(f)$ by noting that $\mathrm{Im}(f) = \mathrm{Span}(\{f(b_1), \ldots, f(b_n)\})$.

Bases for images and kernels

- Consider the homomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x, 0)$.
- A basis for $\mathrm{Ker}(f)$ is $\{(0, 1)\}$.
- A basis for $\mathrm{Im}(f)$ is $\{(1, 0)\}$.

Bases for images and kernels

- Consider the homomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x + y, 2x + 2y)$.
- A basis for $\mathrm{Ker}(f)$ is $\{(1, -1)\}$.
- A basis for $\mathrm{Im}(f)$ is $\{(1, 2)\}$.

Kernels

- Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.
- In this case $\mathrm{Ker}(f_A)$ is trivial.
- A basis for $\mathrm{Im}(f_A)$ is $\{(1,3),(2,4)\}$.