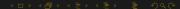
# Math 2130 Linear Algebra Week 10 Images and kernels

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# Today's topics

- Kernels
- 2 Images

#### Definition

Given a homomorphism  $f \colon V \to W$  the  $\mathit{kernel}$  (or  $\mathit{null space}$ ) of f is

$$Ker(f) = \{ v \in V \mid f(v) = 0 \}.$$

■ Note that Ker(f) is a set of vectors from V.

- Let  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ .
- lacksquare The homomorphism  $f_A \colon \mathbb{R}^2 o \mathbb{R}^2$  given by  $\overline{f_A(v)} = Av$  has

$$\operatorname{Ker}(f_A) = \left\{ (x, y) \in \mathbb{R}^2 \mid f_A(x, y) = 0 \right\}$$

$$= \left\{ (x, y) \in \mathbb{R}^2 \mid \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ (x, y) \in \mathbb{R}^2 \mid x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ (0, 0) \right\}.$$

- We ended up with  $Ker(f_A) = \{(0,0)\}$  because the column vectors of A were linearly independent.
- In general, describing  $\mathrm{Ker}(f)$  for some homomorphism f will require us to understand how a linear combination of vectors could equal the zero vector, like when were were checking for linear independence.

- Consider the homomorphism  $f: \mathbb{R}^2 \to \mathbb{R}^2$  given by f(x,y) = (x,0).
- We have that

$$\operatorname{Ker}(f) = \{ (x, y) \in \mathbb{R}^2 \mid f(x, y) = (0, 0) \}$$
$$= \{ (x, y) \in \mathbb{R}^2 \mid (x, 0) = (0, 0) \}$$
$$= \{ (0, y) \in \mathbb{R}^2 \mid y \in \mathbb{R} \}.$$

■ Note that this is the direction in which *f* "squishes" the plane.

- Consider the homomorphism  $f: \mathbb{R}^2 \to \mathbb{R}^2$  given by f(x,y) = (x+y,2x+2y).
- We have that

$$\operatorname{Ker}(f) = \{ (x, y) \in \mathbb{R}^2 \mid f(x, y) = (0, 0) \}$$
$$= \{ (x, y) \in \mathbb{R}^2 \mid (x + y, 2x + 2y) = (0, 0) \}$$
$$= \{ (x, -x) \in \mathbb{R}^2 \mid x \in \mathbb{R} \}.$$

Note that this is again the direction in which f "squishes" the plane.

#### Proposition

Given any homomorphism  $f \colon V \to W$ , we have that  $\operatorname{Ker}(f)$  is a subspace of V.

■ We can show this by using the Subspace Test.

#### Definition

Given a homomorphism  $f \colon V \to W$  the  $\mathit{image}$  (or  $\mathit{range}$ ) of f is

$$\operatorname{Im}(f) = \left\{ f(v) \in W \mid v \in V \right\}.$$

■ Note that Im(f) is a set of vectors from W.

- Let  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ .
- lacksquare The homomorphism  $f_A\colon\mathbb{R}^2 o\mathbb{R}^2$  given by  $f_A(v)=Av$  has

$$\operatorname{Im}(f_A) = \left\{ f_A(x, y) \in \mathbb{R}^2 \mid (x, y) \in \mathbb{R}^2 \right\}$$

$$= \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x, y \in \mathbb{R} \right\}$$

$$= \left\{ x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 4 \end{bmatrix} \in \mathbb{R}^2 \mid x, y \in \mathbb{R}^2 \right\}$$

$$= \mathbb{R}^2.$$

- We ended up with  $\operatorname{Im}(f_A) = \mathbb{R}^2$  because the column vectors of A span  $\mathbb{R}^2$ .
- In general, describing  $\operatorname{Im}(f)$  for some homomorphism f will require us to understand which vectors can be written as a linear combination of a given set of vectors, like when were were checking whether a set of vectors spanned a vector space.

- Consider the homomorphism  $f: \mathbb{R}^2 \to \mathbb{R}^2$  given by f(x,y) = (x,0).
- We have that

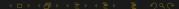
$$\operatorname{Im}(f) = \left\{ f(x, y) \in \mathbb{R}^2 \mid (x, y) \in \mathbb{R}^2 \right\}$$
$$= \left\{ (x, 0) \in \mathbb{R}^2 \mid x, y \in \mathbb{R} \right\}$$
$$= \left\{ (x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R} \right\}.$$

 $\blacksquare$  This is all the stuff we obtain by plugging vectors into f.

- Consider the homomorphism  $f: \mathbb{R}^2 \to \mathbb{R}^2$  given by f(x,y) = (x+y,2x+2y).
- We have that

$$\operatorname{Im}(f) = \left\{ f(x, y) \in \mathbb{R}^2 \mid (x, y) \in \mathbb{R}^2 \right\}$$
$$= \left\{ (x + y, 2x + 2y) \in \mathbb{R}^2 \mid x, y \in \mathbb{R} \right\}$$
$$= \left\{ (x, 2x) \in \mathbb{R}^2 \mid x \in \mathbb{R} \right\}.$$

Note that this is again all the stuff we obtain by plugging vectors into f.



### Proposition

Given any homomorphism  $f \colon V \to W$ , we have that  $\mathrm{Im}(f)$  is a subspace of W.

■ We can show this by using the Subspace Test.