Math 2130
Linear Algebra
Week 10
Images and kernels

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Today's topics

1. Kernels
2. Images

Kernels

Definition

Given a homomorphism $f\colon V \to W$ the *kernel* (or *null space*) of f is

$$\mathrm{Ker}(f) = \{\, v \in V \mid f(v) = 0 \,\}\,.$$

- Note that $\mathrm{Ker}(f)$ is a set of vectors from V.

Kernels

- Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.
- The homomorphism $f_A\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f_A(v) = Av$ has

$$\begin{aligned} \operatorname{Ker}(f_A) &= \{\, (x, y) \in \mathbb{R}^2 \mid f_A(x, y) = 0 \,\} \\ &= \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \\ &= \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \\ &= \{(0, 0)\}\,. \end{aligned}$$

Kernels

- We ended up with $\mathrm{Ker}(f_A) = \{(0,0)\}$ because the column vectors of A were linearly independent.
- In general, describing $\mathrm{Ker}(f)$ for some homomorphism f will require us to understand how a linear combination of vectors could equal the zero vector, like when were were checking for linear independence.

Kernels

- Consider the homomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x, 0)$.
- We have that

$$\begin{aligned}
\mathrm{Ker}(f) &= \{\, (x, y) \in \mathbb{R}^2 \mid f(x, y) = (0, 0) \,\} \\
&= \{\, (x, y) \in \mathbb{R}^2 \mid (x, 0) = (0, 0) \,\} \\
&= \{\, (0, y) \in \mathbb{R}^2 \mid y \in \mathbb{R} \,\}.
\end{aligned}$$

- Note that this is the direction in which f “squishes” the plane.

Kernels

- Consider the homomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x + y, 2x + 2y)$.
- We have that

$$\begin{aligned} \operatorname{Ker}(f) &= \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; f(x, y) = (0, 0) \right\} \\ &= \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; (x + y, 2x + 2y) = (0, 0) \right\} \\ &= \left\{ (x, -x) \in \mathbb{R}^2 \;\middle|\; x \in \mathbb{R} \right\}. \end{aligned}$$

- Note that this is again the direction in which f "squishes" the plane.

Kernels

Proposition

Given any homomorphism $f\colon V \to W$, *we have that* $\mathrm{Ker}(f)$ *is a subspace of* V.

- We can show this by using the Subspace Test.

Images

Definition

Given a homomorphism $f\colon V \to W$ the *image* (or *range*) of f is

$$\mathrm{Im}(f) = \{\, f(v) \in W \mid v \in V \,\}.$$

- Note that $\mathrm{Im}(f)$ is a set of vectors from W.

Images

- Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.
- The homomorphism $f_A \colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f_A(v) = Av$ has

$$\begin{aligned} \operatorname{Im}(f_A) &= \left\{ f_A(x, y) \in \mathbb{R}^2 \;\middle|\; (x, y) \in \mathbb{R}^2 \right\} \\ &= \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x, y \in \mathbb{R} \right\} \\ &= \left\{ x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 4 \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x, y \in \mathbb{R}^2 \right\} \\ &= \mathbb{R}^2. \end{aligned}$$

Images

- We ended up with $\mathrm{Im}(f_A) = \mathbb{R}^2$ because the column vectors of A span $\mathbb{R}^2$.
- In general, describing $\mathrm{Im}(f)$ for some homomorphism f will require us to understand which vectors can be written as a linear combination of a given set of vectors, like when were were checking whether a set of vectors spanned a vector space.

Images

- Consider the homomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x, 0)$.
- We have that

$$\begin{aligned} \mathrm{Im}(f) &= \left\{\, f(x, y) \in \mathbb{R}^2 \;\middle|\; (x, y) \in \mathbb{R}^2 \,\right\} \\ &= \left\{\, (x, 0) \in \mathbb{R}^2 \;\middle|\; x, y \in \mathbb{R} \,\right\} \\ &= \left\{\, (x, 0) \in \mathbb{R}^2 \;\middle|\; x \in \mathbb{R} \,\right\}. \end{aligned}$$

- This is all the stuff we obtain by plugging vectors into f.

Images

- Consider the homomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x + y, 2x + 2y)$.
- We have that

$$\begin{aligned}
\operatorname{Im}(f) &= \left\{\, f(x, y) \in \mathbb{R}^2 \;\middle|\; (x, y) \in \mathbb{R}^2 \,\right\} \\
&= \left\{\, (x + y, 2x + 2y) \in \mathbb{R}^2 \;\middle|\; x, y \in \mathbb{R} \,\right\} \\
&= \left\{\, (x, 2x) \in \mathbb{R}^2 \;\middle|\; x \in \mathbb{R} \,\right\}.
\end{aligned}$$

- Note that this is again all the stuff we obtain by plugging vectors into f.

Images

Proposition

Given any homomorphism $f\colon V \to W$*, we have that* $\mathrm{Im}(f)$ *is a subspace of* W*.*

- We can show this by using the Subspace Test.