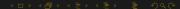
# Math 2130 Linear Algebra Week 10 Homomorphisms

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## Today's topics

Homomorphisms

#### Definition

Given vector spaces V and W a function  $T\colon V\to W$  is said to be a homomorphism when for all  $v_1,v_2\in V$  and all scalars k we have that

- $\blacksquare$  (additivity)  $T(v_1+v_2)=T(v_1)+T(v_2)$  and
- (homogeneity)  $T(kv_1) = kT(v_1)$ .
- Isomorphisms are homomorphisms that are also bijections.
- Homomorphisms are also called *linear transformations* or linear maps.

#### Lemma

For any vector spaces V and W and any function  $f: V \to W$  the following are equivalent:

- $f(v_1 + v_2) = f(v_1) + f(v_2)$  and f(cv) = cf(v)
- $(c_1v_1 + c_2v_2) = c_1f(v_1) + c_2f(v_2)$
- $f(c_1v_1 + \dots + c_nv_n) = c_1f(v_1) + \dots + c_nf(v_n)$
- We saw this before when we talked about dimension and isomorphisms.
- The conditions in the above lemma are all equivalent to f being a homomorphism, since condition (1) is the definition of being a homomorphism.

■ This tells us that if  $f: \mathbb{R}^2 \to \mathbb{R}^3$  is homomorphism with f(1,0)=(1,3,5) and f(0,1)=(2,4,6) then we can compute

$$f(2,3) = f(2(1,0) + 3(0,1))$$

$$= 2f(1,0) + 3f(0,1)$$

$$= 2(1,3,5) + 3(2,4,6)$$

$$= (8,18,28).$$

- Every homomorphism  $f: V \to W$  is determined by what it does to a basis for V.
- Let  $\{b_1,\ldots,b_n\}$  be a basis for V and note that each vector  $v\in V$  can be written uniquely as

$$v = c_1b_1 + c_2b_2 + \dots + c_nb_n$$

for some  $c_1, c_2, \ldots, c_n \in \mathbb{R}$ .

It follows that

$$f(v) = f(c_1b_1 + c_2b_2 + \dots + c_nb_n) = c_1f(b_1) + c_2f(b_2) + \dots + c_nf(b_n),$$

so knowing the  $f(b_i)$  tells us how to compute f(v) for any  $v \in V$ .



- This also means that if  $g: V \to W$  is another homomorphism and  $g(b_i) = f(b_i)$  for each basis vector  $b_i$  then f = g.
- This is because

$$f(v) = c_1 f(b_1) + c_2 f(b_2) + \dots + c_n f(b_n)$$
  
=  $c_1 g(b_1) + c_2 g(b_2) + \dots + c_n g(b_n)$   
=  $g(v)$ ,

so f and g are the same function.

- On the other hand, we can make a homomorphism  $f \colon V \to W$  by choosing  $f(b_i) \in W$  for each basis element  $b_i$ .
- Supposing that we are given these choices, we can define

$$f(v) = f(c_1b_1 + c_2b_2 + \dots + c_nb_n) = c_1f(b_1) + c_2f(b_2) + \dots + c_nf(b_n).$$

■ This only makes sense as a definition of a function because  $\{b_1, b_2, \ldots, b_n\}$  is a basis, so every vector in V can be written as a unique linear combination of the  $b_i$ .

■ To see that f is actually a homomorphism, no matter what we choose for the values of  $f(b_i)$ , first observe that

$$f(v+u) = f((c_1b_1 + \dots + c_nb_n) + (d_1b_1 + \dots + d_nb_n))$$

$$= f((c_1 + d_1)b_1 + \dots + (c_n + d_n)b_n)$$

$$= (c_1 + d_1)f(b_1) + \dots + (c_n + d_n)f(b_n)$$

$$= (c_1f(b_1) + \dots + c_nf(b_n)) + (d_1f(b_1) + \dots + d_nf(b_n))$$

$$= f(v) + f(u).$$

■ The demonstration that f(sv) = sf(v) is similar.

- We can now describe all homomorphisms between finite-dimensional vector spaces.
- Since every n-dimensional vector space is isomorphic to  $\mathbb{R}^n$ , it suffices to describe homomorphisms from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ .
- Claim: Every homomorphism  $f: \mathbb{R}^n \to \mathbb{R}^m$  is of the form  $f_A(v) = Av$  for some  $A \in \operatorname{Mat}_{m \times n}$ .

- Remember our example of  $f: \mathbb{R}^2 \to \mathbb{R}^3$  with f(1,0) = (1,3,5) and f(0,1) = (2,4,6).
- We want a matrix  $A \in \operatorname{Mat}_{3 \times 2}$  so that  $f_A(v) = f(v)$ .
- Make f(1,0) the first column of A and make f(0,1) the second column of A.
- We have that  $f_A(v) = f(v)$  when  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$  since

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}.$$

