Math 2130
Linear Algebra
Week 10
Homomorphisms

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Today's topics

1. Homomorphisms

Homomorphisms

Definition

Given vector spaces V and W a function $T\colon V \to W$ is said to be a *homomorphism* when for all $v_1, v_2 \in V$ and all scalars k we have that

1. (additivity) $T(v_1 + v_2) = T(v_1) + T(v_2)$ and
2. (homogeneity) $T(kv_1) = kT(v_1)$.

- Isomorphisms are homomorphisms that are also bijections.
- Homomorphisms are also called *linear transformations* or *linear maps*.

Homomorphisms

Lemma

For any vector spaces V and W and any function $f\colon V \to W$ the following are equivalent:

1. $f(v_1 + v_2) = f(v_1) + f(v_2)$ *and* $f(cv) = cf(v)$
2. $f(c_1 v_1 + c_2 v_2) = c_1 f(v_1) + c_2 f(v_2)$
3. $f(c_1 v_1 + \cdots + c_n v_n) = c_1 f(v_1) + \cdots + c_n f(v_n)$

- We saw this before when we talked about dimension and isomorphisms.
- The conditions in the above lemma are all equivalent to f being a homomorphism, since condition (1) is the definition of being a homomorphism.

Homomorphisms

- This tells us that if $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ is homomorphism with $f(1,0) = (1,3,5)$ and $f(0,1) = (2,4,6)$ then we can compute

$$\begin{aligned} f(2,3) &= f(2(1,0) + 3(0,1)) \\ &= 2f(1,0) + 3f(0,1) \\ &= 2(1,3,5) + 3(2,4,6) \\ &= (8,18,28). \end{aligned}$$

Homomorphisms

- Every homomorphism $f\colon V \to W$ is determined by what it does to a basis for V.
- Let $\{b_1, \ldots, b_n\}$ be a basis for V and note that each vector $v \in V$ can be written uniquely as

$$v = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n$$

  for some $c_1, c_2, \ldots, c_n \in \mathbb{R}$.
- It follows that

$$f(v) = f(c_1 b_1 + c_2 b_2 + \cdots + c_n b_n) = c_1 f(b_1) + c_2 f(b_2) + \cdots + c_n f(b_n),$$

  so knowing the $f(b_i)$ tells us how to compute $f(v)$ for any $v \in V$.

Homomorphisms

- This also means that if $g \colon V \to W$ is another homomorphism and $g(b_i) = f(b_i)$ for each basis vector b_i then $f = g$.
- This is because

$$\begin{aligned} f(v) &= c_1 f(b_1) + c_2 f(b_2) + \cdots + c_n f(b_n) \\ &= c_1 g(b_1) + c_2 g(b_2) + \cdots + c_n g(b_n) \\ &= g(v), \end{aligned}$$

so f and g are the same function.

Homomorphisms

- On the other hand, we can make a homomorphism $f\colon V \to W$ by choosing $f(b_i) \in W$ for each basis element b_i.
- Supposing that we are given these choices, we can define

$$f(v) = f(c_1 b_1 + c_2 b_2 + \cdots + c_n b_n) = c_1 f(b_1) + c_2 f(b_2) + \cdots + c_n f(b_n).$$

- This only makes sense as a definition of a function because $\{b_1, b_2, \ldots, b_n\}$ is a basis, so every vector in V can be written as a unique linear combination of the b_i.

Homomorphisms

- To see that f is actually a homomorphism, no matter what we choose for the values of $f(b_i)$, first observe that

$$\begin{aligned} f(v+u) &= f((c_1 b_1 + \cdots + c_n b_n) + (d_1 b_1 + \cdots + d_n b_n)) \\ &= f((c_1 + d_1) b_1 + \cdots + (c_n + d_n) b_n) \\ &= (c_1 + d_1) f(b_1) + \cdots (c_n + d_n) f(b_n) \\ &= (c_1 f(b_1) + \cdots + c_n f(b_n)) + (d_1 f(b_1) + \cdots + d_n f(b_n)) \\ &= f(v) + f(u). \end{aligned}$$

- The demonstration that $f(sv) = sf(v)$ is similar.

Homomorphisms

- We can now describe all homomorphisms between finite-dimensional vector spaces.
- Since every n-dimensional vector space is isomorphic to $\mathbb{R}^n$, it suffices to describe homomorphisms from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- Claim: Every homomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is of the form $f_A(v) = Av$ for some $A \in \mathrm{Mat}_{m\times n}$.

Homomorphisms

- Remember our example of $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ with $f(1,0) = (1,3,5)$ and $f(0,1) = (2,4,6)$.
- We want a matrix $A \in \mathrm{Mat}_{3\times 2}$ so that $f_A(v) = f(v)$.
- Make $f(1,0)$ the first column of A and make $f(0,1)$ the second column of A.
- We have that $f_A(v) = f(v)$ when $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ since

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}.$$