Universal algebra and lattice theory
Lecture 7
Complete lattices

Charlotte Aten

2020 December 23

Today's topics

- Motivation
- Definition of a complete lattice
- Examples of complete lattices
- Subuniverse and congruence lattices
- Complete sublattices
- Congruence lattices of groups
- Congruence lattices of lattices

Motivation

- Consider the lattice $\mathbf{L} \coloneqq (\mathrm{Sb}(\mathbb{N}), \cap, \cup)$.
- Given any $\mathscr{X} \subset \mathrm{Sb}(\mathbb{N})$ we have that $\bigcup \mathscr{X} = \bigcup_{X \in \mathscr{X}} X$ belongs to $\mathrm{Sb}(\mathbb{N})$.
- Moreover, $\sup(\mathscr{X}) = \bigcup \mathscr{X}$.
- We think of $\bigcup \mathscr{X}$ as the «infinite join» of the members of $\mathscr{X}$.

Motivation

- Instead consider the lattice $\mathbf{L} := (\mathrm{Sb}_\omega(\mathbb{N}), \cap, \cup)$ where $\mathrm{Sb}_\omega(\mathbb{N})$ consists of all finite subsets of $\mathbb{N}$.
- Let $[n] := \{1, 2, \ldots, n\}$ and take $\mathscr{X} := \{\, [n] \mid n \in \mathbb{N} \,\}$.
- The collection $\mathscr{X}$ has no upper bound in $\mathbf{L}$, much less a least upper bound.
- Thus, $\sup(\mathscr{X})$ does not exist in $\mathbf{L}$.

Definition of a complete lattice

Definition (Complete lattice)

A lattice $\mathbf{L}$ is said to be *complete* when for every $X \subset L$ we have that both $\sup(X)$ and $\inf(X)$ exist.

- We define $\bigvee X := \sup(X)$ and $\bigwedge X := \inf(X)$.
- When $X := \{x_i\}_{i \in I}$ we define $\bigvee_{i \in I} x_i := \sup(X)$ and $\bigwedge_{i \in I} x_i := \inf(X)$.

Examples of complete lattices

- The lattice $(\mathrm{Sb}(\mathbb{N}), \cap, \cup)$ is complete.
- The lattice $(\mathrm{Sb}_\omega(\mathbb{N}), \cap, \cup)$ is not complete.
- The lattice $(\mathrm{Sb}_\omega(\mathbb{N}) \cup \{\mathbb{N}\}, \cap, \cup)$ is complete.
- The lattice $(\mathbb{N}, \min, \max)$ is not complete.
- The lattice $(\mathbb{N} \cup \{\infty\}, \min, \max)$ is complete.
- Note that $(\mathbb{R}, \min, \max)$ is not complete, contrary to the language used in analysis and topology.
- The lattice $(\mathbb{R} \cup \{-\infty, \infty\}, \cap, \cup)$ is complete, however.

Subuniverse and congruence lattices

- At long last we will make lattices out of the subuniverses and congruences of algebras.
- In order to do this, we use the following result.

Proposition

If $\mathbf{P}$ *is a poset in which* $\inf(X)$ *exists for each* $X \subset P$ *then* $\mathbf{P}$ *is a complete lattice.*

Subuniverse and congruence lattices

Proposition

If $\mathbf{P}$ is a poset in which $\inf(X)$ exists for each $X \subset P$ then $\mathbf{P}$ is a complete lattice.

Proof.

We already have that $\mathbf{P}$ has arbitrary infima so it remains to show that $\mathbf{P}$ has arbitrary suprema. Given $X \subset P$ we must produce $\sup(X)$ within $\mathbf{P}$. Take

$$Y := \{\, y \in P \mid y \gg X \,\}$$

and define $a := \inf(Y)$. Given any $x \in X$ we have that $x \leq y$ for each $y \in Y$, so $x \ll Y$. Since a is the greatest among the lower bounds of Y we have $x \leq a$. It follows that $a \in Y$ and is the least of all upper bounds for X. □

Subuniverse and congruence lattices

The following corollary tells us that the subuniverses and congruences of any algebra form complete lattices.

Corollary

Given an algebra $\mathbf{A}$ *we have that* $\mathbf{Sub}(\mathbf{A}) := (\mathrm{Sub}(\mathbf{A}), \subset)$ *and* $\mathbf{Con}(\mathbf{A}) := (\mathrm{Con}(\mathbf{A}), \subset)$ *are complete lattices.*

Proof.

We already know that $\mathrm{Sub}(\mathbf{A})$ and $\mathrm{Con}(\mathbf{A})$ are closed under taking arbitrary intersections, which give our arbitrary infima. There is one point I swept under the rug in a previous talk though: How do we compute $\bigcap \varnothing$? $\square$

Subuniverse and congruence lattices

Corollary

Given an algebra $\mathbf{A}$ *we have that* $\mathbf{Sub}(\mathbf{A}) := (\mathrm{Sub}(\mathbf{A}), \subset)$ *and* $\mathbf{Con}(\mathbf{A}) := (\mathrm{Con}(\mathbf{A}), \subset)$ *are complete lattices.*

Proof.

If we're being really careful then when we compute an intersection $\bigcap \mathscr{X}$ we should always specify that $\mathscr{X} \subset \mathrm{Sb}(A)$ for some set A. We then define

$$\bigcap \mathscr{X} := \{\, a \in A \mid (\forall X \in \mathscr{X})(a \in X) \,\},$$

which yields $\bigcap \varnothing = A$. □

Subuniverse and congruence lattices

- Recall that Ore had a program during the 1930s where lattices became the central objects of study in all of mathematics.
- One of the shortcomings of this approach is that it was not clear how to extract all properties of an object from a corresponding lattice.
- For example, consider the cyclic groups $\mathbf{C}_2$ and $\mathbf{C}_3$.
- We have that

$$\mathbf{Sub}(\mathbf{C}_2) \cong \mathbf{Sub}(\mathbf{C}_3) \cong \mathbf{Con}(\mathbf{C}_2) \cong \mathbf{Con}(\mathbf{C}_3) \cong \mathbf{2}.$$

Subuniverse and congruence lattices

- We have that

$$\mathbf{Sub}(\mathbf{C}_2) \cong \mathbf{Sub}(\mathbf{C}_3) \cong \mathbf{Con}(\mathbf{C}_2) \cong \mathbf{Con}(\mathbf{C}_3) \cong \mathbf{2}.$$

- In the 1920s Ada Rottlaender studied the problem of distinguishing groups by their subgroup lattices using only those isomorphisms which respect conjugation.
- She found that even under this stricter condition there were still nonisomorphic pairs of groups with isomorphic subgroup lattices.

Complete sublattices

We have another corollary of our earlier proposition.

Corollary

Given a set A we have that $\mathbf{Eq}(A) := (\mathrm{Eq}(A), \subset)$ is a complete lattice.

We know that $\mathbf{Eq}(A)$ supports taking arbitrary joins, but how do we actually compute them? Arbitrary meets are easy because in $\mathbf{Eq}(A)$ we have that $\bigwedge \Theta = \bigcap \Theta$ for any $\Theta \subset \mathrm{Eq}(A)$.

Complete sublattices

Proposition

Given a set A *and* $\Theta \subset \mathrm{Eq}(A)$ *we have in* $\mathbf{Eq}(A)$ *that*

$$\bigvee \Theta = 0_A \cup \bigcup \{ \theta_1 \circ \theta_2 \circ \cdots \circ \theta_k \mid k \in \mathbb{N} \text{ and } (\forall i \leq k)(\theta_i \in \Theta) \} .$$

Proof sketch: Let the left- and right-hand-sides be α and β, respectively. Argue that β is an equivalence relation similarly to how we gave an explicit construction of $\mathrm{Cg}^{\mathbf{A}}(\nu)$ previously. It follows that $\alpha \subset \beta$. To show that $\beta \subset \alpha$ note that given $\theta_1, \ldots, \theta_k \in \Theta$ we have that $\theta_1 \circ \cdots \circ \theta_k \subset \alpha \circ \cdots \circ \alpha = \alpha$.

Complete sublattices

Definition (Complete sublattice)

Give a complete lattice $\mathbf{L}$ and a sublattice $\mathbf{M}$ of $\mathbf{L}$ we say that $\mathbf{M}$ is a *complete sublattice* of $\mathbf{L}$ when for each $X \subset M$ we have that $\bigvee X$ and $\bigwedge X$ (as computed in $\mathbf{L}$) are elements of M.

- It is possible for complete lattices to have sublattices which are incomplete and vice versa.
- Consider that $(\mathrm{Sb}_\omega(\mathbb{N}) \cup \{\mathbb{N}\}, \subset)$ is a complete lattice which is a sublattice of the complete lattice $(\mathrm{Sb}(\mathbb{N}), \subset)$ but it is not a complete sublattice.

Complete sublattices

We have some standard examples of complete sublattices available to us.

Theorem

Given an algebra $\mathbf{A}$ *we have that* $\mathbf{Con}(\mathbf{A})$ *is a complete sublattice of* $\mathbf{Eq}(A)$. *Moreover, if* $\mathbf{B}$ *is a reduct of* $\mathbf{A}$ *then* $\mathbf{Con}(\mathbf{A})$ *is a complete sublattice of* $\mathbf{Con}(\mathbf{B})$.

Congruence lattices of groups

We finish today by giving two classic results on the congruence lattices of groups and of lattices.

Theorem (Dedekind, 1900)

The congruence lattice of a group is modular.

Proof.

Note that if α and β are group congruences and $(x, y) \in \alpha \circ \beta$ then there is some z so that $x \; \alpha \; z \; \beta \; y$. It follows that

$$x = (xz^{-1}z) \; \beta \; (xz^{-1}y) \; \alpha \; (zz^{-1}y) = y$$

so $(x, y) \in \beta \circ \alpha$. We find that $\alpha \circ \beta = \beta \circ \alpha$. In this situation we say that α and β *permute* and have that $\alpha \vee \beta = \alpha \circ \beta$. □

Congruence lattices of groups

We finish today by giving two classic results on the congruence lattices of groups and of lattices.

Theorem (Dedekind, 1900)

The congruence lattice of a group is modular.

Proof.

Suppose that α, β, and γ are congruences with $\gamma \subset \alpha$. We must show the nontrivial containment

$$\alpha \wedge (\beta \vee \gamma) \subset (\alpha \wedge \beta) \vee \gamma.$$

Given $(x, y) \in \alpha \wedge (\beta \vee \gamma)$ we have some z so that $x \; \beta \; z \; \gamma \; y$ and since $\gamma \subset \alpha$ we have that $z \; \alpha \; y \; \alpha \; x$. Thus, $x \; (\alpha \wedge \beta) \; z \; \gamma \; y$. □

Congruence lattices of lattices

We finish today by giving two classic results on the congruence lattices of groups and of lattices.

Theorem (Funayama and Nakayama, 1942)

The congruence lattice of a lattice is distributive.

- The congruences of a lattice don't generally commute so this argument takes a little more work. The majority terms we discussed previously when looking at distributivity are very helpful here.
- Recall that every distributive lattice is modular, so the congruence lattices of lattices are more constrained than the congruence lattices of groups.