Universal algebra and lattice theory
Week 3
The distributive and modular laws

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Today's topics

- Definition of distributive and modular lattices
- Relationship between distributivity and modularity
- Dualizing distributivity
- Two very special nondistributive lattices
- Dedekind's result on modularity
- Birkhoff's result on distributivity
- An aside about graph theory

Definition of distributive and modular lattices

Today we introduce two of the main algebraic properties of interest for lattices.

Definition (Distributive lattice)

We say that a lattice $\mathbf{L}$ is *distributive* when $\mathbf{L}$ satisfies

$$x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z).$$

We actually always have

$$x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z)$$

so if we want to check that a lattice is distributive it suffices to show that

$$x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z).$$

Definition of distributive and modular lattices

Today we introduce two of the main algebraic properties of interest for lattices.

Definition (Modular lattice)

We say that a lattice $\mathbf{L}$ is *modular* when for all $y \in L$ we have that

$$z \leq x \text{ implies } x \wedge (y \vee z) = (x \wedge y) \vee z.$$

We actually always have that $z \leq x$ implies

$$x \wedge (y \vee z) \geq (x \wedge y) \vee z$$

so if we want to check that a lattice is modular it suffices to show that $z \leq x$ implies

$$x \wedge (y \vee z) \leq (x \wedge y) \vee z.$$

Relationship between distributivity and modularity

Proposition

Every distributive lattice is modular.

Proof.

Suppose that $\mathbf{L}$ is a lattice with $x, y, z \in L$ such that $z \leq x$. We have that

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

by distributivity. Since $z \leq x$ we have that $x \wedge z = z$ so

$$x \wedge (y \vee z) = (x \wedge y) \vee z,$$

as desired. □

Dualizing distributivity

A distributive lattice is one for which $\wedge$ distributes over $\vee$. What can we say about lattices for which $\vee$ distributes over $\wedge$?

Dualizing distributivity

Proposition

A lattice $\mathbf{L}$ *is distributive if and only if* $\mathbf{L}$ *satisfies*

$$x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z).$$

Proof.

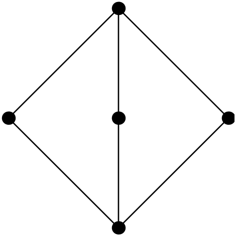
Suppose that $\mathbf{L}$ is distributive. Given $x, y, z \in L$ define $a := x \vee y$. Observe that

$$\begin{aligned}(x \vee y) \wedge (x \vee z) &= a \wedge (x \vee z) = (a \wedge x) \vee (a \wedge z) \\ &= x \vee ((x \vee y) \wedge z) = x \vee (x \wedge z) \vee (y \wedge z) \\ &= x \vee (y \wedge z).\end{aligned}$$

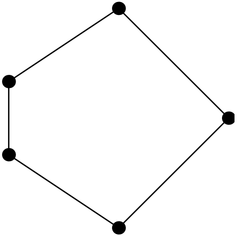
The argument in the other direction is identical. □

Two very special nondistributive lattices

- You should examine all nonempty lattices of order at most 4.
- There are 5 such lattices, up to isomorphism.
- They are all distributive.
- There are exactly two nondistributive lattices of order 5, which we call $\mathbf{M}_3$ and $\mathbf{N}_5$.
- We have that $\mathbf{M}_3$ is modular and $\mathbf{N}_5$ is not.



$\mathbf{M}_3$



$\mathbf{N}_5$

Dedekind's result on modularity

Theorem (Dedekind (1900))

Take $\mathbf{L}$ *to be a lattice. The following are equivalent.*

(a) $\mathbf{L}$ *is modular*

(b) $\mathbf{L}$ *satisfies* $((x \wedge z) \vee y) \wedge z \approx (x \wedge z) \vee (y \wedge z)$

(c) $\mathbf{L}$ *has no sublattice isomorphic to* $\mathbf{N}_5$

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- We show that (a) implies (b).
- Take $x, y, z \in L$ and define $c := x \wedge z$.
- Since $c \leq z$ we have by modularity that $z \wedge (y \vee c) = (z \wedge y) \vee c$.
- This shows that $z \wedge (y \vee (x \wedge z)) = (z \wedge y) \vee (x \wedge z)$.

Dedekind's result on modularity

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(c) $\mathbf{L}$ *has no sublattice isomorphic to* $\mathbf{N}_5$

- We show that (b) implies (c) by proving the contrapositive.
- Suppose that $\mathbf{L}$ has a sublattice isomorphic to $\mathbf{N}_5$ labeled so that $0 < x < z < 1$ and $0 < y < 1$.
- One can verify that this violates the identity in (b).

Dedekind's result on modularity

Theorem (Dedekind (1900))

Take $\mathbf{L}$ to be a lattice. The following are equivalent.

(a) $\mathbf{L}$ *is modular*

(b) $\mathbf{L}$ *satisfies* $((x \wedge z) \vee y) \wedge z \approx (x \wedge z) \vee (y \wedge z)$

(c) $\mathbf{L}$ *has no sublattice isomorphic to* $\mathbf{N}_5$

- We show that (c) implies (a) by proving the contrapositive.
- Suppose that $\mathbf{L}$ is not modular. We must show that $\mathbf{L}$ has a sublattice isomorphic to $\mathbf{N}_5$.
- By assumption there are elements $a, b, c \in L$ with $a \geq c$ so that $a \wedge (b \vee c) > (a \wedge b) \vee c$.
- The desired sublattice has

$$a \wedge b < c \vee (a \wedge b) < a \wedge (b \vee c) < b \vee c$$

and $a \wedge b < b < b \vee c$.

Dedekind's result on modularity

Theorem (Dedekind (1900))

Take $\mathbf{L}$ to be a lattice. The following are equivalent.

(a) $\mathbf{L}$ *is modular*

(b) $\mathbf{L}$ *satisfies* $((x \wedge z) \vee y) \wedge z \approx (x \wedge z) \vee (y \wedge z)$

(c) $\mathbf{L}$ *has no sublattice isomorphic to* $\mathbf{N}_5$

- Even though modularity was not originally defined by an identity, part (b) shows that it could have been.
- This means that homomorphic images, sublattices, and products of modular lattices are all modular, as well.
- Since $\mathbf{L}$ contains a copy of $\mathbf{N}_5$ if and only if $\mathbf{L}^\partial$ does, we have that $\mathbf{L}$ is modular if and only if $\mathbf{L}^\partial$ is.
- We refrain from giving an example here, but not all identities which hold in $\mathbf{L}$ necessarily hold in $\mathbf{L}^\partial$, so modularity is special in this regard.

Birkhoff's result on distributivity

Theorem (Birkhoff)

Take $\mathbf{L}$ *to be a lattice. The following are equivalent.*

(a) $\mathbf{L}$ *is distributive*

(b) $\mathbf{L}$ *satisfies*
$(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$

(c) $\mathbf{L}$ *has no sublattice isomorphic to* $\mathbf{N}_5$ *or* $\mathbf{M}_3$

Birkhoff's result on distributivity

- The left- and right-hand-sides of the identity in (b) are particularly noteworthy.
- Define $m_1(x, y, z) := (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ and $m_2(x, y, z) := (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$.
- For $i \in \{1, 2\}$ we have in any lattice that

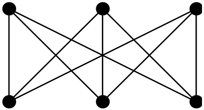
$$m_i(x, x, y) \approx m_i(x, y, x) \approx m_i(y, x, x) \approx x.$$

- A term like m_1 or m_2 which satisfies the above identities is called a *majority term*.

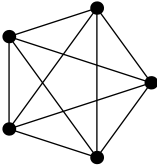
An aside about graph theory

Theorem (Kuratowski)

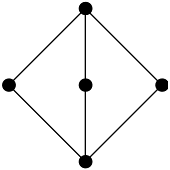
A finite graph is planar if and only if it does not contain a subdivision of either K_5 or $K_{3,3}$ as a subgraph.



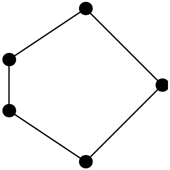
$K_{3,3}$



K_5



$\mathbf{M}_3$



$\mathbf{N}_5$