Universal algebra and lattice theory
Week 3
Posets and lattices

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Today's topics

- Motivation
- Some history
- Posets
- Lattices
- Isotone maps and homomorphisms
- Isotone maps and continuous maps
- The lattice of open sets
- Lattices in probability

Motivation

Recall the definition of a lattice we gave before.

Definition (Lattice)

A *lattice* is an algebra $\mathbf{L} := (L, \wedge, \vee)$ such that $(L, \wedge)$ and $(L, \vee)$ are semilattices and the identities

$$x \wedge (x \vee y) \approx x \text{ and } x \vee (x \wedge y) \approx x$$

hold.

Motivation

- Lattices are going to serve as interesting examples of algebras which don't look much like groups or rings.
- At the same time, understanding lattices will help us with the theory of general algebras.
- We will see on another day how to make collections of congruences, subuniverses, and varieties into lattices.
- Lattice theory has deep ties to many other areas of math, including combinatorics, topology, and probability.

Some history

- George Boole introduced what are now called *Boolean algebras* (which are special kinds of lattices) in the nineteenth century.
- Alfred North Whitehead first used the expression «universal algebra» in his 1898 book «A Treatise on Universal Algebra», which included both groups and Boolean algebras.
- Richard Dedekind, as we previously remarked, worked with lattices of subgroups around the year 1900.

Some history

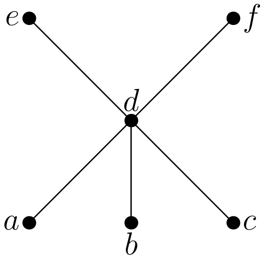
- Lattice theory became an established discipline in its own right during the 1930s and 1940s.
- Garrett Birkhoff published «On the Structure of Abstract Algebras» in 1935, establishing universal algebra as a branch of mathematics.
- Birkhoff used lattice-theoretic ideas in his paper. In 1940 he published a book on lattice theory.
- Øystein Ore referred to lattices as «structures» and led a short-lived program during the 1930s where lattices were hailed as the single unifying concept for all of mathematics.
- During this period Saunders Mac Lane studied algebra under Ore's advisement. Mac Lane went on to become one of the founders of category theory.

Posets

- Although we defined lattices as algebras previously, it turns out that orderings on sets are going to be very relevant here.
- We say that a binary relation θ on a set A is *antisymmetric* when for all $x, y \in A$ we have that $x \; \theta \; y$ and $y \; \theta \; x$ implies that $x = y$.
- A *partial ordering* of a set P is a binary relation on P which is reflexive, transitive, and antisymmetric.
- We usually denote a partial order by the symbol $\leq$.
- We refer to $\mathbf{P} := (P, \leq)$ as a *poset*.
- For example, $(\mathbb{N}, \leq)$ is a poset with the usual definition of $\leq$ for natural numbers.

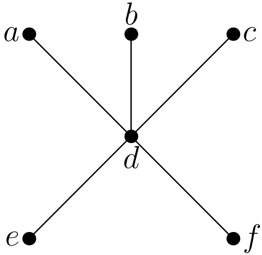
Posets

- In order to depict a poset we may use a *Hasse diagram*, which is a graph whose vertices correspond to the elements of the poset and whose edges indicate the ordering.
- The Hasse diagram of the poset on $\{a, b, c, d, e, f\}$ with $a < d < e$, $b < d < f$, and $c < d$ is depicted below.



Posets

- Given a poset $\mathbf{P} := (P, \sigma)$ we define the *dual* of $\mathbf{P}$ to be $\mathbf{P}^{\partial} := (P, \sigma^{\smile})$.
- Note that the Hasse diagram of $\mathbf{P}^{\partial}$ is just the Hasse diagram of $\mathbf{P}$ upside-down.



Lattices

- Given a poset $\mathbf{P} := (P, \leq)$ we denote by $a \ll X$ the statement «for all $x \in X$ we have that $a \leq x$».
- In the case that $a \ll X$ we say that a is a *lower bound* for X.
- We say that a lower bound a of X is the *greatest lower bound* (or *infimum*) of X when for all $p \in P$ we have that if $p \ll X$ then $p \leq a$.
- We can similarly define *upper bound* and *least upper bound* (or *supremum*).

Lattices

We are finally ready to define a lattice (again).

Definition (Lattice)

A *lattice* is a poset $\mathbf{P} := (P, \leq)$ in which every pair of elements of P has a supremum and infimum.

Of course we already have a definition of a lattice as an algebra.

Definition (Lattice)

A *lattice* is an algebra $\mathbf{L} := (L, \wedge, \vee)$ such that $(L, \wedge)$ and $(L, \vee)$ are semilattices and the identities

$$x \wedge (x \vee y) \approx x \text{ and } x \vee (x \wedge y) \approx x$$

hold.

Lattices

- It only makes sense to have these two different definitions of a lattice if they're somehow equivalent.
- Given a lattice (poset) $(P, \leq)$ we can define an algebra $(P, \wedge, \vee)$ where

$$x \wedge y := \inf(\{x, y\}) \text{ and } x \vee y := \sup(\{x, y\}).$$

- This algebra $(P, \wedge, \vee)$ is always a lattice (algebra).
- There is also an inverse mapping taking each lattice (algebra) to a lattice (poset).
- Given a lattice (algebra) $(P, \wedge, \vee)$ we can define a poset $(P, \leq)$ where we set $x \leq y$ when $x = x \wedge y$.
- This poset $(P, \leq)$ is always a lattice (poset).

Isotone maps and homomorphisms

We consider those functions which respect poset orderings.

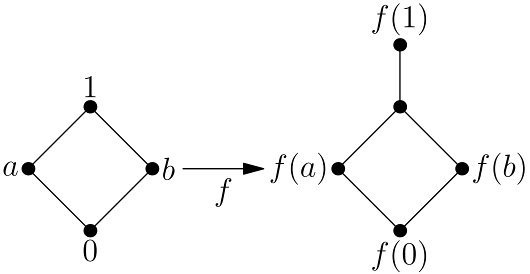
Definition (Isotone map)

Given posets $\mathbf{P} := (P, \leq^{\mathbf{P}})$ and $\mathbf{Q} := (Q, \leq^{\mathbf{Q}})$ we say that a function $f\colon P \to Q$ is *isotone* when for all $x, y \in P$ we have that $x \leq^{\mathbf{P}} y$ implies $f(x) \leq^{\mathbf{Q}} f(y)$.

We write $f\colon \mathbf{P} \to \mathbf{Q}$ in order to indicate that $f\colon P \to Q$ is an isotone map from $\mathbf{P}$ to $\mathbf{Q}$.

Isotone maps and homomorphisms

- When $\mathbf{P}$ and $\mathbf{Q}$ are lattices and $f\colon P \to Q$ is a homomorphism (of algebras) we have that f is isotone.
- Not all isotone maps between lattices are homomorphisms.



Isotone maps and homomorphisms

Proposition

Given a bijective, isotone map $f\colon \mathbf{P} \to \mathbf{Q}$ *we have that if* $f^{-1}\colon Q \to P$ *is isotone then* f *is a lattice isomorphism.*

Proof.

We show that f is a homomorphism, so we must show that for any $a, b \in P$ we have $f(a \wedge^{\mathbf{P}} b) = f(a) \wedge^{\mathbf{Q}} f(b)$. Take $c := a \wedge^{\mathbf{P}} b$. We know that $c \ll^{\mathbf{P}} \{a, b\}$ so by isotonicity we have that $f(c) \ll^{\mathbf{Q}} \{f(a), f(b)\}$. It remains to show that $f(c)$ is the greatest among these lower bounds. Given $x \ll^{\mathbf{Q}} \{f(a), f(b)\}$ we use that f^{-1} is isotone to see that $f^{-1}(x) \ll^{\mathbf{P}} \{a, b\}$, which implies that $f^{-1}(x) \leq^{\mathbf{P}} c$ and hence $x \leq^{\mathbf{Q}} f(c)$. An identical argument works for $\vee$. □

Isotone maps and continuous maps

- We can associate to any poset a topology in the following way.
- Given a poset $\mathbf{P} := (P, \leq)$ we say that $D \subset P$ is a *downset* of $\mathbf{P}$ when for each $x \in D$ and each $y \in P$ we have that $y \leq x$ implies that $y \in D$.
- We denote by $\mathrm{Dn}(\mathbf{P})$ the collection of all downsets of $\mathbf{P}$.
- We have that $\mathrm{Dn}(\mathbf{P})$ is actually a topology on P.
- Moreover, a function $f \colon P \to Q$ is an isotone map from $\mathbf{P}$ to $\mathbf{Q}$ if and only if f is a continuous map from $(P, \mathrm{Dn}(\mathbf{P}))$ to $(Q, \mathrm{Dn}(\mathbf{Q}))$.
- In particular, this means that each homomorphism of lattices is a continuous map between the corresponding topological spaces.

The lattice of open sets

- We have just seen how to make spaces from posets (including lattices), but we can also produce lattices from topological spaces.
- Given a topological space $\mathbf{T} := (T, \tau)$ the algebra $(\tau, \cap, \cup)$ is a lattice.
- This is one of the observations that leads to the study of locale theory (or, more humorously, pointless topology).

Lattices in probability

- Lattices also appear in measure-theoretic probability theory.
- Recall that a σ-*algebra* on a set X is a $\Sigma \subset \mathrm{Sb}(X)$ which contains X, is closed under complementation, and is closed under taking countable unions.
- It follows from De Morgan's laws for sets that any σ-algebra is also closed under countable intersections.
- Given a σ-algebra Σ we find that $(\Sigma, \cap, \cup)$ is a lattice.
- Actually, $(\Sigma, \cap, \cup)$ is a special kind of lattice. It is bounded, complemented, and countably complete. We'll discuss these properties of lattices on subsequent days.