Universal algebra and lattice theory
Week 2
Homomorphisms, subalgebras, and products

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Today's topics

- Preparatory work: eliminating some indices
- Homomorphisms, monoids, and groups
- Subalgebras and subuniverses
- Products
- The operators $\mathbf{H}$, $\mathbf{S}$, and $\mathbf{P}$
- Generating subalgebras

Preparatory work: eliminating some indices

- We previously noted that most of our basic concepts only make sense for algebras of the same similarity type $\rho\colon I \to \mathbb{W}$.
- If we fix our index set to be a collection $\mathcal{F}$ of operation symbols and we fix a similarity type $\rho\colon \mathcal{F} \to \mathbb{W}$ then we can express an algebra as

$$\mathbf{A} := (A, \mathcal{F}^{\mathbf{A}})$$

  where each member $f^{\mathbf{A}} \in \mathcal{F}^{\mathbf{A}}$ is a $\rho(f)$-ary operation on A.
- Note that a particular $f \in \mathcal{F}$ is just taken as an abstract symbol with a specified arity $\rho(f)$, while $f^{\mathbf{A}}$ is actually a function.

Preparatory work: eliminating some indices

- For example, we can focus our attention on (the similarity type of) rings by fixing operation symbols a, m, n, and z of arities 2, 2, 1, and 0, respectively.
- Given a ring $\mathbf{R}$ and some $a, b \in R$ we would then write $m^{\mathbf{R}}(a, b)$ to indicate the product of a and b, $n^{\mathbf{R}}(a)$ to indicate the additive inverse of a, and so on.
- When context allows we write $m(a, b)$ rather than $m^{\mathbf{R}}(a, b)$.
- We can use similar notation for infix symbols as well. Thus, $a \cdot^{\mathbf{R}} b$ may be used instead of $m^{\mathbf{R}}(a, b)$. The superscript specifying the algebra may be omitted in this case too, context permitting.

Homomorphisms, monoids, and groups

Recall the definition we already gave for a homomorphism of algebras.

Definition (Homomorphism)

Given algebras $\mathbf{A} := (A, F)$ and $\mathbf{B} := (B, G)$ of the same similarity type $\rho\colon I \to \mathbb{W}$ we say that a function $h\colon A \to B$ is a *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ when for each $i \in I$ and all $a_1, \dots, a_{\rho(i)} \in A$ we have that

$$h(f_i(a_1, \dots, a_{\rho(i)})) = g_i(h(a_1), \dots, h(a_{\rho(i)})).$$

Homomorphisms, monoids, and groups

With our new notation this definition becomes cleaner.

Definition (Homomorphism)

Given algebras $\mathbf{A} := (A, \mathcal{F}^{\mathbf{A}})$ and $\mathbf{B} := (B, \mathcal{F}^{\mathbf{B}})$ we say that a function $h \colon A \to B$ is a *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ when for each $f \in \mathcal{F}$ of arity n and all $a_1, \ldots, a_n \in A$ we have that

$$h(f^{\mathbf{A}}(a_1, \ldots, a_n)) = f^{\mathbf{B}}(h(a_1), \ldots, h(a_n)).$$

Homomorphisms, monoids, and groups

- We give some more terminology pertaining to homomorphisms.
- We refer to injective homomorphisms as *embeddings*.
- When $h\colon \mathbf{A} \to \mathbf{B}$ is a surjective homomorphism we say that $\mathbf{B}$ is a *homomorphic image* of $\mathbf{A}$.
- A bijective homomorphism is said to be an *isomorphism*.
- A homomorphism $h\colon \mathbf{A} \to \mathbf{A}$ is called an *endomorphism*. The set of all endomorphisms of $\mathbf{A}$ is denoted by $\mathrm{End}(\mathbf{A})$
- An isomorphism $h\colon \mathbf{A} \to \mathbf{A}$ is called an *automorphism*. The set of all automorphisms of $\mathbf{A}$ is denoted by $\mathrm{Aut}(\mathbf{A})$.

Homomorphisms, monoids, and groups

- If $g\colon \mathbf{A} \to \mathbf{B}$ and $h\colon \mathbf{B} \to \mathbf{C}$ are homomorphisms then $h \circ g$ is also a homomorphism.
- Given an algebra $\mathbf{A}$ we have that

  $$\mathbf{End}(\mathbf{A}) := (\mathrm{End}(\mathbf{A}), \circ, \mathrm{id}_A)$$

  is a monoid and

  $$\mathbf{Aut}(\mathbf{A}) := (\mathrm{Aut}(\mathbf{A}), \circ, \_^{-1}, \mathrm{id}_A)$$

  is a group.
- We refer to an algebra whose universe consists of a single element as a *trivial* algebra.
- We call an algebra $\mathbf{A}$ *rigid* when $\mathbf{Aut}(\mathbf{A})$ is trivial.

Subalgebras and subuniverses

Using our new notation for introducing algebras we can rewrite our old definition of a subalgebra.

Definition (Subalgebra)

Given algebras $\mathbf{A} := (A, \mathcal{F}^{\mathbf{A}})$ and $\mathbf{B} := (B, \mathcal{F}^{\mathbf{B}})$ we say that $\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$ when $B \subset A$ and for each $f \in \mathcal{F}$ of arity n we have that $f^{\mathbf{B}} = f^{\mathbf{A}}|_{B^n}$.

Subalgebras and subuniverses

We will often want to intersect two subalgebras in order to obtain another subalgebra. In order to do this, we make use of the following concept.

Definition (Subuniverse)

Given an algebra $\mathbf{A}$ we say that $B \subset A$ is a *subuniverse* of $\mathbf{A}$ when B is the universe of a subalgebra $\mathbf{B}$ of $\mathbf{A}$.

The collection of all subuniverses of an algebra $\mathbf{A}$ is denoted by $\mathrm{Sub}(\mathbf{A})$. We always have that $A \in \mathrm{Sub}(\mathbf{A})$.

Digression: empty algebras

- We say that an algebra whose universe is $\varnothing$ is an *empty* algebra.
- The basic operations of such an algebra are all empty functions.
- An empty algebra of a particular signature $\rho\colon \mathcal{F} \to \mathbb{W}$ can only exist if for each $f \in \mathcal{F}$ we have that $\rho(f) \neq 0$.
- Some authors do not allow empty algebras. The discussion of the merits of accepting empty algebras or not is mostly beyond the scope of this lecture.
- Note that the empty set is a subuniverse of any algebra whose basic operations contain no nullary operations.

Products

Just as we did with homomorphisms and subalgebras, we can also rewrite our definition of products using our new notation.

Definition (Product)

Given a sequence $\left\{\mathbf{A}_j := (A_j, \mathcal{F}^{\mathbf{A}_j})\right\}_{j\in J}$ of algebras we define the *product* of $\{\mathbf{A}_j\}$ to be

$$\prod_{j\in J} \mathbf{A}_j := (A, \mathcal{F}^{\mathbf{A}})$$

where $A := \prod_{j\in J} A_j$ and for each $f \in \mathcal{F}$ of arity n we specify that $f^{\mathbf{A}} \colon A^n \to A$ is given by

$$f^{\mathbf{A}}(\{a_{1,j}\}_{j\in J}, \ldots, \{a_{n,j}\}_{j\in J}) := \left\{f^{\mathbf{A}_j}(a_{1,j}, \ldots, a_{n,j})\right\}_{j\in J}.$$

Products

- When a product $\prod_{j \in J} \mathbf{A}_j$ is indexed over the set $J = \{1, 2, \ldots, k\}$ for some $k \in \mathbb{N}$ we write

$$\mathbf{A}_1 \times \mathbf{A}_2 \times \cdots \times \mathbf{A}_k.$$

- In this case we think of elements of the product as tuples $(a_1, a_2, \ldots, a_k)$ where $a_j \in A_j$.
- In the simplest nontrivial case, we can take the direct product $\mathbf{A}_1 \times \mathbf{A}_2$. Given an operation symbol f of arity n we have that

$$\begin{aligned} &f^{\mathbf{A}_1 \times \mathbf{A}_2}((a_{1,1}, a_{1,2}), (a_{2,1}, a_{2,2}) \ldots, (a_{n,1}, a_{n,2})) \\ &\quad = (f^{\mathbf{A}_1}(a_{1,1}, a_{2,1}, \ldots, a_{n,1}), f^{\mathbf{A}_2}(a_{1,2}, a_{2,2}, \ldots, a_{n,2})). \end{aligned}$$

Products

- We define $\mathbf{A}^I := \prod_{i \in I} \mathbf{A}_i$ where $\mathbf{A}_i := \mathbf{A}$ for each $i \in I$. We call $\mathbf{A}^I$ the I^{th} *direct power* of $\mathbf{A}$.
- We can think of elements of $\mathbf{A}^I$ as functions from I to A. The operations of $\mathbf{A}^I$ act componentwise on these functions according to our previous definition.
- When $I = \{1, 2, \ldots, k\}$ for some $k \in \mathbb{N}$ we write $\mathbf{A}^k$ rather than $\mathbf{A}^I$. We write $\mathbf{A}^0$ rather than $\mathbf{A}^\varnothing$.
- You should convince yourself that $\mathbf{A}^0$ is a trivial algebra for any $\mathbf{A}$ and that $\mathbf{A}^1 \cong \mathbf{A}$.
- For a familiar example, consider the ring of real functions

$$(\mathbb{R}, +, \cdot, -, 0)^{\mathbb{R}}.$$

The operators $\mathbf{H}$, $\mathbf{S}$, and $\mathbf{P}$

As we progress in our study of universal algebra we will often be concerned with the following three operators, which produce new classes of algebras from old ones.

Definition

Given a class $\mathcal{K}$ of similar algebras we define

- $\mathbf{H}(\mathcal{K})$ to be the class of all homomorphic images of members of $\mathcal{K}$,
- $\mathbf{S}(\mathcal{K})$ to be the class of all algebras which are isomorphic to a subalgebra of a member of $\mathcal{K}$, and
- $\mathbf{P}(\mathcal{K})$ to be the class of all algebras which are isomorphic to a direct product of members of $\mathcal{K}$.

The operators $\mathbf{H}$, $\mathbf{S}$, and $\mathbf{P}$

- Note that $\mathbf{H}$, $\mathbf{S}$, and $\mathbf{P}$ are defined so that each of the classes $\mathbf{H}(\mathcal{K})$, $\mathbf{S}(\mathcal{K})$, and $\mathbf{P}(\mathcal{K})$ are closed under isomorphic images, no matter what $\mathcal{K}$ is.
- Given an operator $\mathbf{O}$ taking classes of similar algebras to other classes of algebras we say that a class $\mathcal{K}$ is *closed under* $\mathbf{O}$ when $\mathbf{O}(\mathcal{K}) \subset \mathcal{K}$.
- A *variety* is a class of similar algebras $\mathcal{K}$ which is closed under $\mathbf{H}$, $\mathbf{S}$, and $\mathbf{P}$.
- Universal algebra was initiated as a mathematical discipline during the 1930s. Since the 1970s the subject has increasingly focused on varieties. Our treatment will parallel this development.

Generating subalgebras

We said previously that we defined subuniverses so that we could take intersections of subalgebras. The follow proposition moves us in that direction.

Proposition

Given an algebra $\mathbf{A}$ and a collection of subuniverses $\mathcal{S}$ of $\mathbf{A}$ we have that $\bigcap \mathcal{S}$ is a subuniverse of $\mathbf{A}$.

Generating subalgebras

We also would like to consider the smallest subuniverse containing some specified elements of an algebra.

Definition (Subuniverse generated by a set)

Given an algebra $\mathbf{A}$ and $X \subset A$ we define the *subuniverse of* $\mathbf{A}$ *generated by* X to be

$$\mathrm{Sg}^{\mathbf{A}}(X) := \bigcap \{\, U \in \mathrm{Sub}(\mathbf{A}) \mid X \subset U \,\}\,.$$

The previous proposition tells us that $\mathrm{Sg}(X)$ is indeed a subuniverse of $\mathbf{A}$. Note that $\mathrm{Sg}(X)$ must contain X.

Generating subalgebras

Instead of taking this «top-down» viewpoint using intersections we can give a «bottom-up» description of $\mathrm{Sg}^{\mathbf{A}}(X)$ by taking unions.

Theorem

Given an algebra $\mathbf{A} := (A, F)$ *and* $X \subset A$ *we define* $X_0 := X$ *and for each* $n \in \mathbb{N}$ *we define* X_n *to be*

$$X_{n-1} \cup \{\, f(a_1, \ldots, a_k) \mid f \in F,\ \rho(f) = k, \text{ and } a_1, \ldots, a_k \in X_{n-1} \,\}\,.$$

We have that $\mathrm{Sg}^{\mathbf{A}}(X) = \bigcup_{n \in \mathbb{W}} X_n$.

Generating subalgebras

- We sketch the proof that $\mathrm{Sg}^{\mathbf{A}}(X) = \bigcup_{n \in \mathbb{W}} X_n$.
- Take $Y := \bigcup_{n \in \mathbb{W}} X_n$. It suffices to show that $\mathrm{Sg}^{\mathbf{A}}(X) \subset Y$ and that $Y \subset \mathrm{Sg}^{\mathbf{A}}(X)$.
- In order to show that $\mathrm{Sg}^{\mathbf{A}}(X) \subset Y$ we can show that Y is a subuniverse of $\mathbf{A}$ containing X, but this is evident from the construction of Y.
- In order to show that $Y \subset \mathrm{Sg}^{\mathbf{A}}(X)$ we use induction on n to show that each of the X_n are contained in $\mathrm{Sg}(X)$.

Generating subalgebras

- A consequence of the preceding result is that if $a \in \mathrm{Sg}^{\mathbf{A}}(X)$ for some $a \in A$ then there is some finite $Y \subset X$ such that $a \in \mathrm{Sg}(Y)$.
- We say that an algebra $\mathbf{A}$ is *finitely generated* when there exists some finite $Y \subset A$ such that $A = \mathrm{Sg}^{\mathbf{A}}(Y)$.