Multiplayer Rock-Paper-Scissors

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2018

RPS as a Magma

We will view the game of RPS as a magma. We let $A := \{r, p, s\}$ and define a binary operation $f\colon A^2 \to A$ where $f(x, y)$ is the winning item among $\{x, y\}$.

	r	p	s
r	r	p	r
p	p	p	s
s	r	s	s

Selection Games

A *selection game* is a game consisting of a collection of items A, from which a fixed number of players n each choose one, resulting in a tuple $a \in A^n$, following which the round's winners are those who chose $f(a)$ for some fixed rule $f\colon A^n \to A$. RPS is a selection game, and we can identify each such game with an *n-ary magma* $\mathbf{A} := (A, f)$.

Properties of RPS

The game RPS is

1. conservative,
2. essentially polyadic,
3. strongly fair, and
4. nondegenerate.

These are the properties we want for a multiplayer game, as well.

Properties of RPS: Conservativity

We say that an operation $f\colon A^n \to A$ is *conservative* when for any $a_1, \ldots, a_n \in A$ we have that $f(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$. We say that **A** is conservative when each round has at least one winning player.

Properties of RPS: Essential Polyadicity

We say that an operation $f\colon A^n \to A$ is *essentially polyadic* when there exists some $g\colon \mathrm{Sb}(A) \to A$ such that for any $a_1, \ldots, a_n \in A$ we have $f(a_1, \ldots, a_n) = g(\{a_1, \ldots, a_n\})$. We say that **A** is essentially polyadic when a round's winning item is determined solely by which items were played, not taking into account which player played which item or how many players chose a particular item.

Properties of RPS: Strong Fairness

Let A_k denote the members of A^n which have k distinct components for some $k \in \mathbb{N}$. We say that f is *strongly fair* when for all $a, b \in A$ and all $k \in \mathbb{N}$ we have $\left|f^{-1}(a) \cap A_k\right| = \left|f^{-1}(b) \cap A_k\right|$. We say that **A** is strongly fair when each item has the same chance of being the winning item when exactly k distinct items are chosen for any $k \in \mathbb{N}$.

Properties of RPS: Nondegeneracy

We say that f is *nondegenerate* when $|A| > n$. In the case that $|A| \leq n$ we have that all members of $A_{|A|}$ have the same set of components. If $\mathbf{A}$ is essentially polyadic with $|A| \leq n$ it is impossible for $\mathbf{A}$ to be strongly fair unless $|A| = 1$.

Variants with More Items

The French version of RPS adds one more item: the well. This game is not strongly fair but is conservative and essentially polyadic. The recent variant Rock-Paper-Scissors-Spock-Lizard is conservative, essentially polyadic, strongly fair, and nondegenerate.

	r	p	s	w
r	r	p	r	w
p	p	p	s	p
s	r	s	s	w
w	w	p	w	w

	r	p	s	v	l
r	r	p	r	v	r
p	p	p	s	p	l
s	r	s	s	v	s
v	v	p	v	v	l
l	r	l	s	l	l

Result for Two-Player Games

The only "valid" RPS variants for two players use an odd number of items.

Theorem

Let $\mathbf{A}$ *be a selection game with* $n = 2$ *which is essentially polyadic, strongly fair, and nondegenerate and let* $m := |A|$*. We have that* $m \neq 1$ *is odd. Conversely, for each odd* $m \neq 1$ *there exists such a selection game.*

RPS Magmas

Definition (RPS magma)

Let $\mathbf{A} := (A, f)$ be an n-ary magma. When $\mathbf{A}$ is conservative, essentially polyadic, strongly fair, and nondegenerate we say that $\mathbf{A}$ is an RPS *magma*. When $\mathbf{A}$ is an n-magma of order m with these properties we say that $\mathbf{A}$ is an RPS(m, n) *magma*. We also use RPS and RPS(m, n) to indicate the classes of such magmas.

Result for Multiplayer Games

Theorem

Let **A** *be a selection game with n players and m items which is essentially polyadic, strongly fair, and nondegenerate. For all primes $p \leq n$ we have that $p \nmid m$. Conversely, for each pair (m, n) with $m \neq 1$ such that for all primes $p \leq n$ we have that $p \nmid m$ there exists such a selection game.*

Proof (Forward Direction)

Since **A** is nondegenerate we must have that $m > n$.
Since **A** is strongly fair we must have that $\left|f^{-1}(a) \cap A_k\right| = \left|f^{-1}(b) \cap A_k\right|$ for all $k \in \mathbb{N}$. As the m distinct sets $f^{-1}(a) \cap A_k$ for $a \in A$ partition A_k and are all the same size we require that $m \mid |A_k|$. When $k > n$ we have that $A_k = \varnothing$ and obtain no constraint on m.

Proof (Forward Direction)

When $k \leq n$ we have that A_k is nonempty. As we take **A** to be essentially polyadic we have that $f(x) = f(y)$ for all $x, y \in A_k$ such that $\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\}$. Let B_k denote the collection of unordered sets of k distinct elements of A. Note that the size of the collection of all members $x \in B_k$ such that $\{x_1, \ldots, x_n\} = \{z_1, \ldots, z_k\}$ for distinct $z_i \in A$ does not depend on the choice of distinct z_i. This implies that for a fixed $k \leq n$ each of the m items must be the winner among the same number of unordered sets of k distinct elements in A. We have that $|B_k| = \binom{m}{k}$ so we require that $m \mid |B_k| = \binom{m}{k}$ for all $k \leq n$.

Proof (Forward Direction)

Let

$$d(m,n) := \gcd\left(\left\{ \binom{m}{k} \,\middle|\, 1 \le k \le n \right\}\right).$$

Since $m \mid \binom{m}{k}$ for all $k \le n$ we must have that $m \mid d(m,n)$. Joris, Oestreicher, and Steinig showed that when $m > n$ we have

$$d(m,n) = \frac{m}{\operatorname{lcm}(\{ k^{\varepsilon_k(m)} \mid 1 \le k \le n \})}$$

where $\varepsilon_k(m) = 1$ when $k \mid m$ and $\varepsilon_k(m) = 0$ otherwise. Since we have that $m \mid d(m,n)$ and $d(m,n) \mid m$ it must be that $m = d(m,n)$ and hence

$$\operatorname{lcm}\left(\left\{ k^{\varepsilon_k(m)} \,\middle|\, 1 \le k \le n \right\}\right) = 1.$$

This implies that $\varepsilon_k(m) = 0$ for all $2 \le k \le n$. That is, no k between 2 and n inclusive divides m. This is equivalent to having that no prime $p \le n$ divides m, as desired.

Items as a Function of Players

Our numerical condition also allows us to fix the number of items m and ask how many players n may use that number of items.

Theorem

Given a fixed m there exists an $\mathrm{RPS}(m, n)$ magma if and only if $n < t(m)$ where $t(m)$ is the least prime dividing m.

Algebraic Properties of RPS Magmas

The class RPS is not closed under taking subalgebras. The French variant is a subalgebra of Rock-Paper-Scissors-Spock-Lizard. The class of RPS magmas is as far from being closed under products as possible.

Theorem

Let $\mathbf{A}$ *and* $\mathbf{B}$ *be nontrivial* RPS *n-magmas with* $n > 1$*. The magma* $\mathbf{A} \times \mathbf{B}$ *is not an* RPS *magma.*

This can be done by showing that the product $\mathbf{A} \times \mathbf{B}$ is not conservative.

Current Directions

1. Geometric interpretation as in tournaments.
2. Asymptotics on conservativity.
3. Properties of clones. Note the connection with cyclic/symmetric groups.

Thank you.